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Positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian

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ABSTRACT

In the framework of variational methods, we use a two non-zero critical points theorem to obtain the existence of two positive solutions to Dirichlet boundary value problems for difference equations involving the discrete p-Laplacian operator.

1. Introduction

Difference equations arise in different research fields (for example, computer science, discrete optimization, economics, population genetics, etc.). Many authors have discussed the existence and multiplicity of solutions for difference problems by exploiting various methods from nonlinear analysis, including the method of upper and lower solutions, fixed point theory, Rabinowitz's global bifurcation theorem, Leray–Schauder degree and critical groups (see [5,6,23] and references therein). We refer the reader to [1] and [24] for a general overview on difference equations and related topics.

Recently, many new results have been established by applying variational methods (see [7,9,10,12–21] and references therein). Due to applications in physics, such as non-Newtonian fluid mechanics, turbulence of porous media, positive solutions of p-Laplacian discrete boundary value problems are studied by many authors, see, for instance, [3] and references therein.

Let N be a positive integer, denote with $[1, N]$ the discrete interval $\{1, \ldots, N\}$ and consider the following problem

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$$\begin{cases} -\Delta\left(\phi_p\left(\Delta u\left(k-1\right)\right)\right)+q(k)\phi_p\left(u\left(k\right)\right)=\lambda f\left(k,u\left(k\right)\right) & k\in[1,N], \\ u(0)=u(N+1)=0, \end{cases} \tag{$D_\lambda^{f,q}$}$$

where $f:[1,N]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $\Delta u\left(k-1\right)=u(k)-u(k-1)$ is the forward difference operator, $q(k)\geq 0$ for all $k\in[1,N]$, $\phi_p(s)=|s|^{p-2}s$, $1<p<+\infty$ and λ is a positive real parameter.

In this paper, under suitable assumptions on the nonlinearity f, we obtain the existence of two positive solutions to problem $(D_\lambda^{f,q})$. Our main tool is a two critical points theorem (Theorem 2.1) established in [11]. Such critical point result is an appropriate combination of the local minimum theorem obtained in [8], with the classical and seminal Ambrosetti–Rabinowitz theorem (see [4]). A crucial assumption of the mountain pass theorem is the Palais–Smale condition. It is satisfied in the applications in an infinite dimensional space by requiring a condition on the nonlinear term stronger than p-superlinearity at infinity. In this paper it is proved that the p-superlinearity at infinity of the primitive on the nonlinear datum is sufficient to prove the Palais–Smale condition. It is worth noticing that, here, to obtain the existence of two positive solutions, it is enough to assume only an algebraic condition on the nonlinearity (see condition (3.1) in Theorem 3.1), which is more general than the p-sublinearity at zero. Essentially, the existence of at least two solutions to $(D_\lambda^{f,q})$ is obtained by requiring the p-superlinearity at infinity and the p-sublinearity at zero on the primitive of f (see Corollary 3.1). Moreover, by requiring on the nonlinearity that $f(k,0)\geq 0$, we obtain the existence result of positive solutions by applying a strong maximum principle proved here (see Theorem 2.2), by following the reasoning in [12].

Here, we state a special case of our main result.

Theorem 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that*

$$\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}}=+\infty, \tag{1.1}$$

and

$$\lim_{t\to +\infty}\frac{f(t)}{t^{p-1}}=+\infty.$$

Then, for each $\lambda\in\left]0,\frac{2^p}{pN(N+1)^{p-1}}\sup_{c>0}\frac{c^p}{\max_{|\xi|\leq c}\int_0^{\xi}f(t)\,dt}\right[$*, the problem*

$$\begin{cases} -\Delta\left(\phi_p\left(\Delta u\left(k-1\right)\right)\right)+q(k)\phi_p\left(u\left(k\right)\right)=\lambda f\left(u\left(k\right)\right) & k\in[1,N], \\ u(0)=u(N+1)=0, \end{cases}$$

admits at least two positive solutions.

This theorem extends the result of [14, Theorem 1.1] to discrete p-laplacian. The presence of weight q does not modify the interval of parameters λ.

This paper is organized as follows. In Section 2, some definitions and results on difference equations are collected. An estimation of the equivalence of the norms are provided (see Proposition 2.1). Moreover, Lemma 2.1 and Theorem 2.2 are presented to guarantee, respectively, the Palais–Smale condition and the strong maximum principle in discrete framework related to problem $(D_\lambda^{f,q})$. The abstract critical points theorem (Theorem 2.1) is recalled, and some remarks on classical results for discrete problems are presented.