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Quadratic and cubic harmonic number sums

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A B S T R A C T

This paper develops an approach to evaluation of Euler related sums. The approach is based on simple integral computations. By the approach, we can obtain some closed form representations of sums of quadratic and cubic harmonic numbers and reciprocal binomial coefficients. The given representations are new.

1. Introduction

In this paper, by using the integrals of polylogarithm functions, we discuss the analytic representations of the following type of quadratic and cubic Euler related sums involving harmonic numbers and reciprocal binomial coefficients

$$\sum_{n=1}^{\infty} \frac{H_n \zeta_n(m)}{n^p \binom{n+k}{k}}, \quad \sum_{n=1}^{\infty} \frac{H_n^3}{n^p \binom{n+k}{k}}, \tag{1.1}$$

through harmonic numbers and Riemann zeta functions, where m, p, k are positive integers, H_n and $\zeta_n(m)$ stand for the n-th harmonic number and the n-th generalized harmonic number, which are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j}, \quad \zeta_n(k) = \sum_{j=1}^{n} \frac{1}{j^k}, \quad 1 \le k \in Z. \tag{1.2}$$

On the other hand, we define the n-th generalized alternating harmonic number by

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$$L_n(k) = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^k},\ 1 \le k \in Z. \tag{1.3}$$

In [21], Anthony Sofo gave the following relations for harmonic numbers and alternating harmonic numbers

$$L_n(k) = \zeta_{2[(n+1)/2]-1}(k) - \frac{1}{2^k}\left(\zeta_{[n/2]}(k) + \zeta_{[(n-1)/2]}(k)\right),\ 2 \le k \in Z,$$
$$L_n(1) = H_n - H_{[n/2]},$$

where $[x]$ is the integer part of x.

The polylogarithm function is defined for $|x| \le 1$ by

$$\mathrm{Li}_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \Re(p) > 1. \tag{1.4}$$

If $x = 1$, then the function $\mathrm{Li}_p(x)$ reduces to the Riemann zeta function $\zeta(p)$ which is defined by (see [1,4,5])

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \Re(p) > 1.$$

If $x = -1$, then the function $\mathrm{Li}_p(x)$ reduces to the alternating Riemann zeta function which is defined by

$$\bar{\zeta}(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p},\ \Re(p) \ge 1.$$

The subject of this paper is Euler sums. The classical Euler sums are the infinite sums whose general term is a product of harmonic numbers of index n and a power of n^{-1}. For instance, the classical linear Euler sum is defined by

$$S_{p,q} = \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q}, 2 \le q \in Z. \tag{1.5}$$

Let $\pi = (\pi_1, \ldots, \pi_k)$ be a partition of integer p and $p = \pi_1 + \cdots + \pi_k$ with $\pi_1 \le \pi_2 \le \cdots \le \pi_k$. The classical nonlinear Euler sum of index π, q is defined as follows (see [14])

$$S_{\pi,q} = \sum_{n=1}^{\infty} \frac{\zeta_n(\pi_1)\zeta_n(\pi_2)\cdots\zeta_n(\pi_k)}{n^q}, \tag{1.6}$$

where the quantity $\pi_1 + \cdots + \pi_k + q$ is called the weight, the quantity k is called the degree. Many Euler sums can be expressed in terms of a linear rational combination of Riemann zeta values. For example we know that [2,14]

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^4} = \frac{231}{16}\zeta(7) - \frac{51}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5)$$

and from [25]

$$\sum_{n=1}^{\infty} \frac{H_n \zeta_n(2)}{n^4} = -\frac{51}{16}\zeta(7) + \frac{3}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5).$$