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# Scaling variables and asymptotic profiles for the semilinear damped wave equation with variable coefficients

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ABSTRACT

We study the asymptotic behavior of solutions for the semilinear damped wave equation with variable coefficients. We prove that if the damping is effective, and the nonlinearity and other lower order terms can be regarded as perturbations, then the solution is approximated by the scaled Gaussian of the corresponding linear parabolic problem. The proof is based on the scaling variables and energy estimates.



## 1. Introduction

We consider the Cauchy problem of the semilinear damped wave equation with lower order perturbations

$$\begin{cases} u_{tt} + b(t)u_t = \Delta_x u + c(t) \cdot \nabla_x u + d(t)u + N(u, \nabla_x u, u_t), & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x),\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the coefficients $b$, $c$ and $d$ are smooth, $b$ satisfies

$$b(t) \sim (1+t)^{-\beta}, \quad -1 \le \beta < 1, \tag{1.2}$$

and $c(t) \cdot \nabla_x u, d(t)u, N(u, \nabla_x u, u_t)$ can be regarded as perturbations (the precise assumption will be given in the next section). Also, $\varepsilon$ denotes a small parameter.

Our purpose is to give the asymptotic profile of global solutions to (1.1) with small initial data as time tends to infinity. By the assumption (1.2), the damping is effective, and we can expect that the asymptotic profile of solutions is given by the scaled Gaussian (see (2.7), (2.8) and (2.9)).

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The existence of global solutions and the asymptotic behavior of solutions to damped wave equations have been widely investigated for a long time. Matsumura [27] obtained decay estimates of solutions to the linear damped wave equation

$$u_{tt} - \Delta u + u_t = 0, \tag{1.3}$$

and applied them to nonlinear problems. After that, Yang and Milani [52] showed that the solution of (1.3) has the so-called *diffusion phenomena*, that is, the asymptotic profile of solutions to (1.3) is given by the Gaussian in the $L^\infty$-sense. Marcati and Nishihara [26] and Nishihara [31] gave a more detailed information about the asymptotic behavior of solutions. They found that when $n = 1, 3$, the solution of (1.3) is asymptotically decomposed into the Gaussian and a solution of the wave equation (with an exponentially decaying coefficient) in the $L^p$–$L^q$ sense (see Hosono and Ogawa [12] and Narazaki [30] for $n = 2$ and $n \geq 4$).

For the nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + u_t = N(u), \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), \end{cases} \tag{1.4}$$

there are many results about global existence and asymptotic behavior of solutions (see for example, [13,14,17,19,20,22,32]). In particular, Todorova and Yordanov [41] and Zhang [53] proved that when $N(u) = |u|^p$, the critical exponent of (1.4) is given by $p = 1 + 2/n$. More precisely, they showed that, for initial data satisfying $(u_0, u_1) \in H^{1,0}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and having compact support, if $p > 1+2/n$, then the global solution uniquely exists for small $\varepsilon$; if $p \leq 1 + 2/n$ and $\int_{\mathbb{R}^n} (u_0 + u_1)(x)dx > 0$, then the local-in-time solution blows up in finite time for any $\varepsilon > 0$. The number $1 + 2/n$ is the same as the well-known Fujita exponent, which is the critical exponent of the semilinear heat equation $v_t - \Delta v = v^p$ (see [7]), though the role of the critical exponent is different in the semilinear heat equation and the semilinear damped wave equation. In fact, for the subcritical case $1 < p < 1+2/n$, the solution of the semilinear damped wave equation blows up in finite time under the positive mass condition $\int_{\mathbb{R}^n} (u_0 + u_1)(x)dx > 0$, while all positive solutions blow up in finite time for the semilinear heat equation.

Concerning the asymptotic behavior of the global solution, Hayashi, Kaikina and Naumkin [10] proved that if $N$ satisfies $|N(u)| \leq C|u|^p$ with $p > 1+2/n$, then the unique global solution exists for suitably small data and the asymptotic profile of the solution is given by a constant multiple of the Gaussian. However, they used the explicit formula of the fundamental solution of the linear problem in the Fourier space, and hence, it seems to be difficult to apply their method to variable coefficient cases.

Gallay and Raugel [8] considered the one-dimensional damped wave equation with variable principal term and a constant damping

$$u_{tt} - (a(x)u_x)_x + u_t = N(u, u_x, u_t).$$

They used scaling variables

$$s = \log(t + t_0), \quad y = \frac{x}{\sqrt{t + t_0}}, \tag{1.5}$$

and showed that if $a(x)$ is positive and has the positive limits $\lim_{x\to\pm\infty} a(x) = a_\pm$, then the solution can be asymptotically expanded in terms of the corresponding parabolic equation. Moreover, this expansion can be determined up to the second order. Recently, Takeda [39,40] and Kawakami and Takeda [18] obtained the complete expansion for the linear and nonlinear damped wave equation with constant coefficients.

The wave equation with variable coefficient damping

$$u_{tt} - \Delta u + b(t, x)u_t = 0$$