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Submanifolds with constant scalar curvature in a space form

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We deal with complete submanifolds M^n having constant positive scalar curvature and immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c \in \{1, 0, -1\}$. In this setting, we show that such a submanifold M^n must be either totally umbilical or isometric to a Clifford torus $\mathbb{S}^1\left(\sqrt{1-r^2}\right) \times \mathbb{S}^{n-1}(r)$, when $c = 1$, a circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, when $c = 0$, or a hyperbolic cylinder $\mathbb{H}^1\left(-\sqrt{1+r^2}\right) \times \mathbb{S}^{n-1}(r)$, when $c = -1$. This characterization theorem corresponds to a natural improvement of previous ones due to Alías, García-Martínez and Rigoli [2], Cheng [4] and Guo and Li [6].

1. Introduction

Many authors have approached the problem of characterizing hypersurfaces immersed with constant mean curvature or with constant scalar curvature in a Riemannian space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c. For instance, in the seminal work [5], Cheng and Yau introduced a new self-adjoint differential operator $\square$ acting on smooth functions defined on Riemannian manifolds. As a by-product of such approach they were able to classify closed hypersurfaces M^n with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in $\mathbb{Q}_c^{n+1}$. Later on, Li [8] extended the results due to Cheng and Yau [5] in terms of the squared norm of the second fundamental form of the hypersurface M^n.

In [3], Brasil Jr., Colares and Palmas used the generalized maximum principle of Omori [9] and Yau [13] to characterize complete hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$. In [1], by applying a weak Omori–Yau maximum principle due to Pigola, Rigoli, Setti [10], Alías and García-Martínez studied the behavior of the scalar curvature R of a complete hypersurface immersed with constant mean curvature into

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a real space form $\mathbb{Q}_c^{n+1}$, deriving a sharp estimate for the infimum of R. Afterwards, Alías, García-Martínez and Rigoli [2] obtained another suitable weak maximum principle for complete hypersurfaces with constant scalar curvature in $\mathbb{Q}_c^{n+1}$, and gave some applications of it in order to estimate the norm of the traceless part of its second fundamental form. In particular, they extended the main theorem of [3] for the context of $\mathbb{Q}_c^{n+1}$.

Considering higher codimension, Cheng [4] showed that the totally umbilical sphere $\mathbb{S}^n(r)$, totally geodesic Euclidean space $\mathbb{R}^n$ and generalized cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)$ are the only n-dimensional complete submanifolds with constant scalar curvature and parallel normalized mean curvature vector field (that is, the normalized mean curvature vector field is parallel as a section of the normal bundle) in the Euclidean space $\mathbb{R}^{n+p}$, which satisfy a suitable constrain on the norm of the second fundamental form. Later on, Guo and Li [6] generalized the results of [8] showing that the only closed submanifolds in the unit sphere $\mathbb{S}^{n+p}$ with constant scalar curvature, parallel normalized mean curvature vector field and whose second fundamental form satisfies some appropriate boundedness are the totally umbilical sphere $\mathbb{S}^n(r)$ and the Clifford torus $\mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^{n-1}(r)$.

Motivated by these works, we deal with complete submanifolds M^n having constant positive scalar curvature and immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c\in\{1,0,-1\}$. In this setting, we establish a suitable Simons type formula (cf. Proposition 3.1) and an Omori type maximum principle for the square operator (cf. Lemma 4.5) in order to obtain the following characterization result:

Theorem 1.1. *Let M^n be a complete submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ ($c\in\{1,0,-1\}$ and $n\geq 4$), with constant normalized scalar curvature $R\geq 1$, when $c=1$, and $R>0$, when $c\in\{0,-1\}$. Then*

i. *either $|\Phi|\equiv 0$ and M^n is totally umbilical,*
ii. *or*

$$\sup_M |\Phi|^2 \geq \alpha_{n,c}(R) = \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}.$$

Moreover, assuming in addition that $R>1$ when $c=1$, the equality $\sup_M|\Phi|^2=\alpha_{n,c}(R)$ holds and this supremum is attained at some point of M^n if, and only if, M^n is isometric to a

(a) *Clifford torus $\mathbb{S}^1\left(\sqrt{1-r^2}\right)\times\mathbb{S}^{n-1}(r)\hookrightarrow\mathbb{S}^{n+1}\hookrightarrow\mathbb{S}^{n+p}$, when $c=1$,*
(b) *circular cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)\hookrightarrow\mathbb{R}^{n+1}\hookrightarrow\mathbb{R}^{n+p}$, when $c=0$,*
(c) *hyperbolic cylinder $\mathbb{H}^1\left(-\sqrt{1+r^2}\right)\times\mathbb{S}^{n-1}(r)\hookrightarrow\mathbb{H}^{n+1}\hookrightarrow\mathbb{H}^{n+p}$, when $c=-1$,*

where $r=\sqrt{\dfrac{n-2}{nR}}$.

Here, Φ stands for the traceless part of the second fundamental form of the submanifold M^n. We point out that Theorem 1.1 is a natural extension of Theorems 1 and 2 in [2] for higher codimension and, as well as, it can be regarded as a suitable improvement of the main results of [4] and [6]. The proof of Theorem 1.1 is given in Section 5.

2. Preliminaries

Let M^n be an n-dimensional connected submanifold immersed in a space form $\mathbb{Q}_c^{n+p}$, with constant sectional curvature c. We will make use of the following convention on the range of indices:

$$1\leq A,B,C,\ldots\leq n+p,\quad 1\leq i,j,k,\ldots\leq n \quad\text{and}\quad n+1\leq\alpha,\beta,\gamma,\ldots\leq n+p.$$