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Global existence and boundedness in a Keller–Segel–(Navier–)Stokes system with signal-dependent sensitivity

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A B S T R A C T

In this paper, we consider the following Keller–Segel(–Navier)–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{$\star$}$$

where $\Omega\subset\mathbb{R}^N$ $(N=2,3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\kappa\in\mathbb{R}$ and $\chi(c)$ is assumed to generalize the prototype

$$\chi(c) = \frac{\chi_0}{(1+\mu c)^2},\ c\geq 0.$$

It is proved that i) for $\kappa\neq 0$ and $N=2$ or $\kappa=0$ and $N\in\{2,3\}$, the corresponding initial–boundary problem admits a unique global classical solution which is bounded; ii) for $\kappa\neq 0$ and $N=3$, the corresponding initial–boundary problem possesses at least one global weak solution.

1. Introduction

When *Bacillus subtilis* are suspended in sessile drops of water, populations of such simple individuals exhibit quite colorful collective behavior. More precisely, in colonies of *Bacillus subtilis*, cells will aggregate as plumes, and large-scale fluid motion as well as convection patterns may emerge spontaneously [3,24]. A mathematical model proposed in [24] to describe such process is as follows

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$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(n,c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $n = n(x,t)$ and $c = c(x,t)$ represent the density of cell population and the oxygen concentration, respectively, $u = u(x,t)$ and $P = P(x,t)$ separately stand for the velocity of incompressible fluid and the associated pressure, χ, f, ϕ are given parameter functions and $\kappa \in \mathbb{R}$ denotes the strength of nonlinear fluid convection. The essential modeling hypotheses of (1.1) can be summarized in two respects: on the one hand, bacterial motion is dominated by random diffusion and transport through the fluid, and by chemotactic migration toward increasing gradients of oxygen; on the other hand, the quantity n consumes oxygen and has an influence on the fluid motion by buoyant forces.

During the past years, analytical results on (1.1) seem to concentrate on the issue whether the solutions of corresponding initial–boundary problem are global in time and bounded. In the spatially two-dimensional setting, without any smallness assumptions on either ϕ or on initial data, a unique global classical solution of (1.1) is constructed in [31]. Apart from that, if linear cell diffusion is replaced by porous medium diffusion and χ is supposed to be a chemotactic sensitivity tensor, the work [14] shows that the chemotaxis-Navier–Stokes system (1.1) admits a bounded weak solution on a two-dimensional domain. For the three-dimensional case, global solvability of (1.1) and porous medium-type variants thereof are proved in [32] and [35], respectively. Also in three-dimensional space, when the nonlinear convection term in (1.1) is removed by setting $\kappa = 0$, global weak solutions are established in [31], and more recently the boundedness and asymptotics of solutions to the corresponding chemotaxis-Stokes system (1.1) is proved in the case of porous medium-type diffusion and χ being a chemotactic sensitivity tensor [28].

As in the classical Keller–Segel model where the chemoattractant is produced, rather than consumed, by bacteria, the corresponding chemotaxis-fluid model is then Keller–Segel-fluid system of the form

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(n,c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

In contrast to (1.1), the mathematical analysis of (1.2) is quite fragmentary [4,18,20,21,25]. Among these results, it can be observed that the global boundedness of solutions to the Keller–Segel–Stokes system (1.2) (with $\kappa = 0$) is established when $\chi(n,c)$ is a tensor satisfying some dampening condition with respective to n, i.e.

$$\chi(n,c) \le \frac{\chi_0}{(1+n)^\alpha} \qquad \text{for some } \alpha > 0 \text{ and } \chi_0 > 0 \tag{1.3}$$

in two-dimensional space [25]. It is shown that in the three-dimensional bounded domain, a globally bounded solution exists for the Keller–Segel–Stokes system (1.2) with nonlinear diffusion and logistic source [18]. Notice that the logistic-type cell kinetics is an important blow-up preventing mechanism of (1.2). Indeed, for the three-dimensional Keller–Segel–Stokes system (1.2) with $\chi(n,c) \equiv 1$, the solutions are globally bounded provided that the dampening effect of logistic term is appropriately strong [21]. In addition, for the two-dimensional Keller–Segel–Navier–Stokes system (1.2) (with $\kappa \neq 0$), the bounded result is proved in the presence of logistic source [20].

If the fluid interaction in (1.2) is neglected, i.e. $u \equiv 0$, then (1.2) becomes the following well-known Keller–Segel model