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Area functions characterizations of weighted Bergman spaces

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A B S T R A C T

The aim of this paper is to find weights W in the unit ball of $\mathbb{C}^n$ for which characterization of the area integrals of Bergman spaces $A^p(W)$ holds. The area functions, related to those used to describe Hardy spaces, involve the radial derivative, the complex gradient and the invariant gradient. We extend to certain Bekollé weights the characterization by Z. Chen and W. Ouyang of the Bergman spaces with the classical weights $W(z) = (1 - |z|)^\alpha$ using area functions.

1. Introduction

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball and dv the volume measure on $\mathbb{B}_n \subset \mathbb{R}^{2n}$. We define a weight as a measurable function $W > 0$ in $\mathbb{B}_n$, and for $\alpha \in \mathbb{R}$ we will write $W_\alpha = (1 - |z|^2)^\alpha W$. For $p > 0$ we use the standard notation $L^p(W)$ for the weighted Lebesgue space, with

$$\|f\|_{L^p(W)} = \left(\int_{\mathbb{B}_n} |f|^p W \, dv \right)^{1/p},$$

and the corresponding weighted Bergman space is

$$A^p(W) = L^p(W) \cap hol(\mathbb{B}_n),$$

where $hol(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. For $\alpha > -1$ let dv_α be the measure on $\mathbb{B}_n$ given by

$$dv_\alpha(z) = (1 - |z|^2)^\alpha dv(z),$$

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and consider the Forelli–Rudin Bergman projection P_α which is the orthogonal projection of $L^2(v_\alpha)$ onto $A^2(v_\alpha)$ and is given by

$$P_\alpha f(z) = c_\alpha \int_{\mathbb{B}_n} \frac{f(w)\, dv_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \quad z \in \mathbb{B}_n, \tag{1}$$

for $f \in L^2(v_\alpha)$ and with $c_\alpha = \Gamma(n+\alpha+1)/[n!\Gamma(\alpha+1)]$.

Also consider the operator P^*_α given by

$$P^*_\alpha f(z) = \int_{\mathbb{B}_n} \frac{|f(w)|dv_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}, \quad z \in \mathbb{B}_n.$$

In [3] it was proved a characterization of functions in the Bergman space $A^p(v_\alpha)$ in terms of the area integral functions related to those used to study real-variable Hardy spaces: For $1 < q < \infty$, $r > 0$, consider the following area integral functions

$$A^{r;q}_{\mathcal{R}}(f)(z) = \left(\int_{D(z,r)} |(1-|w|^2)\mathcal{R}f(w)|^q d\tau(w) \right)^{1/q},$$

$$A^{r;q}_{\nabla}(f)(z) = \left(\int_{D(z,r)} |(1-|w|^2)\nabla f(w)|^q d\tau(w) \right)^{1/q},$$

$$A^{r;q}_{\widetilde{\nabla}}(f)(z) = \left(\int_{D(z,r)} |\widetilde{\nabla} f(w)|^q d\tau(w) \right)^{1/q},$$

where $\widetilde{\nabla}$ defined below denotes the invariant gradient operator, $\mathcal{R}f = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}$ is the radial derivative of f, and

$$d\tau(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

is the so-called invariant measure on $\mathbb{B}_n$.

In [3] it was proved that a holomorphic function $f \in hol(\mathbb{B}_n)$ belongs to $A^p(v_\alpha)$, with $p > 0$, is equivalent to one (and hence all) of the conditions $A^{r;q}_{\mathcal{R}}(f) \in L^p(v_\alpha)$, $A^{r;q}_{\nabla}(f) \in L^p(v_\alpha)$, $A^{r;q}_{\widetilde{\nabla}}(f) \in L^p(v_\alpha)$. The purpose of this paper is to extend this result to general weighted Bergman spaces $A^p(W)$. Namely we consider the following properties of $f \in hol(\mathbb{B}_n)$:

h1) $f \in A^p(W)$.
h2) $A^{r;q}_{\mathcal{R}}(f) \in L^p(W)$.
h3) $A^{r;q}_{\nabla}(f) \in L^p(W)$.
h4) $A^{r;q}_{\widetilde{\nabla}}(f) \in L^p(W)$.

The aim of this paper is to find weights W making these properties equivalent. A natural condition to accomplish this will be that W satisfies a Bekolle's condition, namely a continuity of some P^*_β in a Lebesgue space with weight W. The main theorem of the paper is