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Degenerate integrodifferential equations of parabolic type with Robin boundary conditions: L^p-theory

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ABSTRACT

The aim of this paper consists in solving integrodifferential problem of type (1.1)–(1.2) that may degenerate both in space and time. More precisely, $\{M_p(t)\}_{t\in[0,T]}$ is a family of multiplication operators related to a scalar function $m(t,x)$ that may vanish, while $\{L_p(t)\}_{t\in[0,T]}$ is the realization of a family of linear second-order differential operators, with smooth coefficients, $\{\mathcal{L}(t)\}_{t\in[0,T]}$, $\{L_p(t)\}$ being invertible for all $t \in [0,T]$. Moreover, $\{B_p(t,s)\}_{t,s\in[0,T],s\le t}$ is the realization of a family $\{\mathcal{B}(t,s)\}_{t\in[0,T],s\le t}$ of linear second-order differential operators with smooth coefficients. Finally, the scalar functions a and b are such that $1/a$ and b/a are Hölder-continuous with suitable Hölder exponents.

1. Introduction

At the beginning of this paper we want to remember Professor Alfredo Lorenzi, who passed away on November 9th, 2013. He will be greatly missed, both as a dear friend and a brilliant researcher. His enthusiasm, integrity and passion will keep inspiring us throughout our lives.

The present paper is concerned with the initial value problem for the following degenerate – both in space and time – integrodifferential equation of parabolic type

$$\frac{d}{dt}(M_p(t)u(t)) + a(t)L_p(t)u(t) + b(t)\int_0^t B_p(t,s)u(s)ds = f(t), \quad 0 < t \le T, \tag{1.1}$$

$$M_p(0)u(0) = M_p(0)u_0, \tag{1.2}$$

where $L_p(t)$, $1 < p < \infty$, is the realization of a second-order linear elliptic differential operator $\mathcal{L}(t)$ in $L^p(\Omega)$ with the Robin boundary condition, $M_p(t)$ is the multiplication of a function in $L^p(\Omega)$ by some nonnegative

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function $m(\cdot, t)$ and $B_p(t, s)$ is the realization of a linear second-order partial differential operator for each $0 \le s \le t \le T$. Moreover, the reciprocal $1/a$ of the scalar function $a \in C((0, T]) \cap L^1(0, T)$ is Hölder-continuous with a suitable Hölder exponent, while the scalar function b is such that functions $1/a$ and b/a are Hölder-continuous with a suitable Hölder exponent. Therefore, a and b may have some singularity at $t = 0$.

In the previous paper [7] we studied the case $p = 2$ with $a \equiv 1$ and $b \equiv 1$. In this paper we try to extend these results to the case $p \neq 2$ and to problem (1.1)–(1.2), where the pair (a, b) has the properties stated above.

The first – and main – part of this paper deals with the case $a \equiv 1$ and $b \equiv 1$. Our first task consists of constructing the fundamental solution $U(t, s)$, $0 \le s \le t \le T$, to the problem

$$\frac{d}{dt} v(t) + A_p(t)v(t) \ni f(t), \quad 0 < t \le T, \tag{1.3}$$

$$v(0) = v_0 = M_p(0)u_0, \tag{1.4}$$

where $A_p(t) = L_p(t)M_p(t)^{-1}$ is a possibly multivalued operator. This fundamental solution enables us to solve the equation without the integral term:

$$\frac{d}{dt}(M_p(t)u(t)) + L_p(t)u(t) = f(t), \quad 0 < t \le T, \tag{1.5}$$

$$M_p(0)u(0) = M_p(0)u_0. \tag{1.6}$$

The most important part in the construction of the fundamental solution is to establish the following resolvent estimates in the spaces of negative norm:

$$\|M_p(t)(\lambda M_p(t) + \widetilde{L}_p(t))^{-1}\|_{\mathcal{L}(W^{1,p'}(\Omega)^*, W^{1,p'}(\Omega)^*)} \le C|\lambda|^{-\delta_1}, \tag{1.7}$$

$$\|M_p(t)(\lambda M_p(t) + \widetilde{L}_p(t))^{-1}\|_{\mathcal{L}(W^{1,p'}(\Omega)^*, L^p(\Omega))} \le C|\lambda|^{-\delta_2} \tag{1.8}$$

where δ_1, $\delta_2 \in (0, 1]$ are some constants, and $\widetilde{L}_p(t)$ is the extension of $L_p(t)$ to a bounded linear operator from $W^{1,p}(\Omega)$ to $W^{1,p'}(\Omega)^*$ which is defined by

$$a(t; u, v) = (\widetilde{L}_p(t)u, v)_{W^{1,p'}(\Omega)}, \quad u \in W^{1,p}(\Omega),\ v \in W^{1,p'}(\Omega),$$

through the sesquilinear form $a(t; u, v)$ associated with the present Robin boundary problem, the bracket in the right hand side being the pairing between $W^{1,p'}(\Omega)^*$ and $W^{1,p'}(\Omega)$. In case $p = 2$ (1.7) and (1.8) with $\delta_1 = 1$ and $\delta_2 = 1/2$ are easily shown following the method of Example 3.3 in pp. 74–76 of A. Favini and A. Yagi [3], Chap. III in the case of the Dirichlet problem. In case $p \neq 2$ the above method cannot be applied, and an essentially different argument is necessary to establish such estimates. Using the results of the theory of interpolation spaces by A.P. Calderon [1], J.L. Lions and J. Peetre [9] and R. Seeley [11] we show that (1.7) and (1.8) hold with some $\delta_1 < 1$ and $\delta_2 < 1/2$ both dependent on p. However, at the present stage in the construction of the fundamental solution the range of the values of p is limited to the interval $(2, 8/3)$ in case $p > 2$.

We also need some more resolvent estimates in case m satisfies a certain regularity hypothesis called ρ-regularity. These estimates are established in A. Favini, A. Lorenzi, H. Tanabe and A. Yagi [6] under some assumptions on the coefficients. In the present paper we show that they remain valid without these additional assumptions.

Once the fundamental solution is constructed, problem (1.1)–(1.2) can be solved in just the same manner as the previous paper [7].