J. Math. Anal. Appl. 446 (2017) 1167–1182

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An Extension Theorem for convex functions of class $C^{1,1}$ on Hilbert spaces ☆

Daniel Azagra [a], Carlos Mudarra [b,*]

[a] ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040 Madrid, Spain
[b] ICMAT (CSIC-UAM-UC3-UCM), Calle Nicolás Cabrera 13-15, 28049 Madrid, Spain

ARTICLE INFO

Article history:
Received 6 May 2016
Available online 12 September 2016
Submitted by R.M. Aron

Keywords:
Convex function
$C^{1,1}$ function
Whitney extension theorem

ABSTRACT

Let $\mathbb{H}$ be a Hilbert space, $E \subset \mathbb{H}$ be an arbitrary subset and $f : E \to \mathbb{R}$, $G : E \to \mathbb{H}$ be two functions. We give a necessary and sufficient condition on the pair (f, G) for the existence of a *convex* function $F \in C^{1,1}(\mathbb{H})$ such that $F = f$ and $\nabla F = G$ on E. We also show that, if this condition is met, F can be taken so that $\text{Lip}(\nabla F) = \text{Lip}(G)$. We give a geometrical application of this result, concerning interpolation of sets by boundaries of $C^{1,1}$ convex bodies in $\mathbb{H}$. Finally, we give a counterexample to a related question concerning smooth convex extensions of smooth convex functions with derivatives which are not uniformly continuous.

1. Introduction and main results

Throughout this paper $\mathbb{H}$ will be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$. The norm in $\mathbb{H}$ will be denoted by $\| \cdot \|$. By a 1-jet (f, G) on a subset $E \subset \mathbb{H}$ we understand a pair of functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{H}$. Given a 1-jet (f, G) defined on $E \subset \mathbb{H}$, Le Gruyer proved in [5] that a necessary and sufficient condition on the jet (f, G) for having a $C^{1,1}$ extension F to the whole space $\mathbb{H}$ is that

$$\Gamma(f, G, E) := \sup_{x,y \in E} \left(\sqrt{A_{x,y}^2 + B_{x,y}^2} + |A_{x,y}| \right) < \infty,$$

where

$$A_{x,y} = \frac{2(f(x) - f(y)) + \langle G(x) + G(y), y - x \rangle}{\|x - y\|^2} \quad \text{and}$$

☆ D. Azagra was partially supported by Ministerio de Educación, Cultura y Deporte, Programa Estatal de Promoción del Talento y su Empleabilidad en I+D+i, Subprograma Estatal de Movilidad (PR2015-00298). C. Mudarra was supported by Programa Internacional de Doctorado Fundación La Caixa–Severo Ochoa. Both authors are partially supported by MTM2012-34341.
* Corresponding author.
E-mail addresses: azagra@mat.ucm.es (D. Azagra), carlos.mudarra@icmat.es (C. Mudarra).

http://dx.doi.org/10.1016/j.jmaa.2016.09.015
0022-247X/

$$B_{x,y} = \frac{\|G(x) - G(y)\|}{\|x-y\|} \quad \text{for all} \quad x, y \in E, x \neq y.$$

This condition is equivalent to

$$2 \sup_{y \in \mathbb{H}} \sup_{a \neq b \in E} \frac{f(a) - f(b) + \langle G(a), y-a \rangle - \langle G(b), y-b \rangle}{\|a-y\|^2 + \|b-y\|^2} < \infty,$$

and in particular Le Gruyer's Theorem provides a generalization (to the setting of Hilbert spaces) of Glaeser's version of the Whitney Extension Theorem [8,4] for $C^{1,1}$ functions. See also [7] for another generalization of the $C^{1,1}$ version of the Whitney Extension Theorem for functions defined on subsets of the Hilbert space, and for a proof that the Whitney Extension Theorem fails for C^3 functions on Hilbert spaces. We also refer to [6] for a version of the Whitney Extension Theorem for C^1 functions on some Banach spaces (including Hilbert spaces).

Le Gruyer also shows in [5] that the extension F satisfies

$$\operatorname{Lip}(\nabla F) = \Gamma(F, \nabla F, \mathbb{H}) = \Gamma(f, G, E).$$

Our purpose in this paper is to solve an analogous problem for convex functions. In a recent paper [2], by introducing a new condition $(CW^{1,1})$, see Definition 1.1 below, we gave a satisfactory solution to this problem for convex functions of the class $C^{1,1}(\mathbb{R}^n)$ (in fact, for all the classes $C^{1,\omega}(\mathbb{R}^n)$, where ω is a modulus of continuity), with a good control of the Lipschitz constant of the gradient of the extension in terms of that of G, namely $\operatorname{Lip}(\nabla F) \leq c(n) \operatorname{Lip}(G)$, where $c(n)$ only depends on n (but tends to infinity with n); see also [1] for information about related problems of higher order. In this paper we generalize and improve this result for $C^{1,1}$ convex functions defined on an arbitrary Hilbert space, showing in particular that those constants $c(n)$ can all be taken equal to 1. Nevertheless, it must be observed that whereas the proofs in [2] (and of course the proof of the $C^{1,1}$ version of the Whitney Extension Theorem too) are constructive, the proofs of Le Gruyer's Theorem in [5] and of the main result in the present paper (which is strongly inspired by that of Le Gruyer's) are not, as they both rely on an application of Zorn's lemma.

Definition 1.1. We will say that a pair of functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{H}$ defined on a subset $E \subset \mathbb{H}$, satisfies condition $(CW^{1,1})$ on E provided that there exists a constant $M > 0$ with

$$f(x) - f(y) - \langle G(y), x-y \rangle \geq \frac{1}{2M} \|G(x) - G(y)\|^2 \qquad (CW^{1,1})$$

for all $x, y \in E$.

Remark 1.2. If (f, G) satisfies $(CW^{1,1})$ on E, then

$$f(x) \geq f(y) + \langle G(y), x-y \rangle \quad \text{for all} \quad x, y \in E \qquad (1.1)$$

and

$$\sup_{x \neq y,\, x,y \in E} \left\{ \frac{|f(x) - f(y) - \langle G(y), x-y \rangle|}{\|x-y\|^2}, \frac{\|G(x) - G(y)\|}{\|x-y\|} \right\} \leq M. \qquad (1.2)$$

In particular G is M-Lipschitz on E. Moreover, if we further assume that E is open and convex, then f is $C^{1,1}(E)$, convex and $\nabla f = G$ is M-Lipschitz on E.