J. Math. Anal. Appl. 446 (2017) 1196–1218

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of value for differential games with incomplete information and signals on initial states and payoffs

Xiaochi Wu

Laboratoire de Mathématiques de Bretagne Atlantique, CNRS-UMR 6205, Université de Bretagne Occidentale, 6, avenue Victor le Gorgeu, CS 93837, 29238 Brest cedex 3, France

A R T I C L E   I N F O

Article history:
Received 12 February 2016
Available online 20 September 2016
Submitted by J.A. Filar

Keywords:
Differential games
Incomplete information
Signal
Hamilton–Jacobi–Isaacs equation

A B S T R A C T

This article is devoted to proving the existence of the value for a two-players differential game with incomplete information and a signal structure on the data (initial positions, dynamics, and payoffs). Before the game starts, the data are chosen randomly in a finite set. The players are not informed on the exact data but they receive a signal. The main result of the paper says that the value of the game does exist for suitably defined nonanticipative random strategies. To obtain such a result, we introduce and study a new type of second-order double-obstacle Hamilton–Jacobi–Isaacs equation. We also prove that the value is the unique viscosity solution of this partial differential equation.

1. Introduction

The theory of differential games with incomplete information was first introduced in [4], which is a generalization of the two-person zero-sum repeated games with incomplete information of Aumann and Maschler [1]. For a glimpse of the results in the domain of differential games with incomplete information, readers may refer to the following articles [3,5–10,16,19]. Our work consists in investigating a precise structure of information with signals.

The notion of signals is commonly studied in the theory of repeated games with incomplete information, related results for repeated games can be found in, for example, [14,17,22]. Here we investigate the concept of signals for differential games.

Let us now explain our game model in details. Let $t_0 \in [0,1]$, $n \in \mathbb{N}_*$, $I \in \mathbb{N}_*$, and $X_0 = (x_0^i)_{i\in I} \in (\mathbb{R}^n)^I$. (In this paper, we abuse slightly the notation by denoting a finite set and its cardinal by the same symbol. For example, in this case, we set $I = \{1, 2, 3, \ \dots\ I\}$.) We consider a family of I two-person zero-sum differential games $\left(\mathcal{G}_i(t_0, x_0^i)\right)_{i\in I}$. Each differential game $\mathcal{G}_i(t_0, x_0^i)$ is defined by:

- a dynamical system:

E-mail address: Xiaochi.Wu@etudiant.univ-brest.fr.

http://dx.doi.org/10.1016/j.jmaa.2016.09.035

$$\begin{cases} x'(t) = f^i\big(x(t), u(t), v(t)\big), \ u(t) \in U, \ v(t) \in V, \ t \in [t_0, 1], \\ x(t_0) = x_0^i, \end{cases} \tag{P_i}$$

- a payoff function:

$$\mathcal{J}^i(t_0, x_0^i, u, v) := \int_{t_0}^{1} \gamma^i\big(X_s^{t_0,x_0^i,u,v}, u(s), v(s)\big)ds + g^i(X_1^{t_0,x_0^i,u,v}).$$

Here, U, V are some compact metric spaces. The functions $(f^i : \mathbb{R}^n \times U \times V \to \mathbb{R}^n)_{i\in I}$ are supposed to satisfy standard regularity assumptions ensuring that for any $i \in I$, the dynamical system (P_i) admits a unique solution, denoted by $t \mapsto X_t^{t_0,x_0^i,u,v}$. The controls $u : [t_0, 1] \to U$ and $v : [t_0, 1] \to V$ are both Lebesgue measurable. The functions $\big(\gamma^i \in L^1(\mathbb{R}^n \times U \times V)\big)_{i\in I}$ and $(g^i : \mathbb{R}^n \to \mathbb{R})_{i\in I}$ are respectively the running costs and the terminal costs.

Let us consider two finite signal sets $K = \{1, 2, \ \dots \ , K\}$ and $L = \{1, 2, \ \dots \ , L\}$, with $K, L \in \mathbb{N}_*$. Let $h_1 : I \to K$ and $h_2 : I \to L$ be two signal generators. Let $\Delta(I)$ denote the set of all probability measures on I, or equivalently, the I-simplex. For any $p \in \Delta(I)$, we define a differential game $\tilde{\mathcal{G}}(t_0, X_0, p)$ with incomplete information and signals by:

1. Before the game begins, an element $i \in I$ is chosen randomly according to the probability $p \in \Delta(I)$. The signal $k = h_1(i)$ is communicated to Player I but not to Player II, and the signal $l = h_2(i)$ is communicated to Player II but not to Player I.
2. The game $\mathcal{G}_i(t_0, x_0^i)$ is played. More precisely, Player I chooses the control $t \mapsto u(t)$ and aims to minimize $\mathcal{J}^i(t_0, x_0^i, u, v)$. By choosing the control $t \mapsto v(t)$, Player II aims to maximize $\mathcal{J}^i(t_0, x_0^i, u, v)$.

In addition, we suppose that the probability p and the family $\big(\mathcal{G}_i(t_0, x_0^i)\big)_{i\in I}$ are commonly known by the players, and both players observe the actions played by their opponent, i.e., we suppose perfect monitoring in $\mathcal{G}_i(t_0, x_0^i)$, for any $i \in I$.

During the game, each player will try to guess the chosen game $\mathcal{G}_i(t_0, x_0^i)$ by observing his own signal and the actions of his opponent. In the meantime, they will try to hide their own information from their opponent by playing random strategies.

In this paper we investigate a more general structure of signals which includes in a particular case the notion of signals described above. We will describe later on this general structure, which is adapted from the theory of repeated games (cf. [17]).

Differential games with incomplete information without signals investigated in [4,19] can be considered as particular cases of the game $\tilde{\mathcal{G}}(t_0, X_0, p)$.

The goal of this paper is to prove the existence of a value for differential games with incomplete information and signals. For the case without signals, it has been proved in [4,19] that the value of the game exists and this value can be characterized as the unique viscosity solution of a Hamilton–Jacobi–Isaacs equation. But these results can not apply to the case with signals. There are two main difficulties: The first one is to find a suitable notion of nonanticipative random strategies that characterizes the fact that both players choose their strategies according to their own signals. The second one lays on the definition of the value functions over an information space which is much bigger than those of [4,19].

To obtain the wished result, in this paper we will follow the classic scheme of [13]:

- Prove that the upper and lower value functions are viscosity solutions of the same Hamilton–Jacobi–Isaacs equation under the Isaacs' condition introduced in Section 2.
- Prove that this Hamilton–Jacobi–Isaacs equation has a unique viscosity solution.
- Thus the upper and lower value functions must coincide.