



Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for generalized fractional integrals ☆



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ABSTRACT

In this paper we obtain the Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for fractional integrals which generalize the two familiar fractional integrals namely, the Riemann–Liouville and the Hadamard fractional integrals into a single form. We prove that, in most cases, we obtain the Riemann–Liouville and the Hadamard equivalence just by taking limits when a parameter $\rho \rightarrow 1$ and $\rho \rightarrow 0^+$, respectively.

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1. Introduction

The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \rightarrow \mathbb{R}$. The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Let $f : I \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

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$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (1)$$

which is known as the Hermite–Hadamard inequality [11]. In [9], Fejér developed the weighted generalization of the Hermite–Hadamard inequality given below.

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) dx \quad (2)$$

holds, where $g : [a, b] \rightarrow \mathbb{R}$ is non-negative, integrable and symmetric to $(a+b)/2$.

Since then, many researches generalized and extended the two inequalities (1) and (2). For related results, for example, see [8,22–25,32,40–42] and the references therein. In [37], Sarikaya et al. generalized the Hermite–Hadamard type inequalities via Riemann–Liouville fractional integrals. Then in [12], İşcan extended Sarikaya’s results to Hermite–Hadamard–Fejér type inequalities for fractional integrals. Further results involving the two inequalities in question with applications to fractional integrals can be found, for example, in [6,13,37,38] and the references therein.

In [15], the second author introduces an Erdélyi–Kober type fractional integral operator and uses that integral to define a new fractional derivative in [16], which generalizes the Riemann–Liouville and the Hadamard fractional derivatives to a single form and argued that it is not possible to derive the Hadamard equivalence operators from the corresponding Erdélyi–Kober type operators, thus making the new derivative more appropriate for modeling certain phenomena which undergo bifurcation-like behaviors. For further properties of the Erdélyi–Kober operators, the interested reader is referred to, for example, [20,21,36]. According to the literature, the newly defined fractional operators are now known as the *Katugampola fractional integral and derivatives*, respectively [43]. For consistency, we use the same name for those operators in question. It can be shown that the derivatives in question satisfy the fractional derivative criteria (test) given in [19,31]. These operators have applications in fields such as in probability theory [1], theory of inequalities [32,41,44], variational principle [2], numerical analysis [3], and Langevin equations [39]. A Caputo-type modification of the operator in question can be found in [4]. The interested reader is referred, for example, to [5,10,18,26–30,34,35] for further results on these and similar operators. The Mellin transforms of the generalized fractional integrals and derivatives defined in [15] and [16], respectively, are given in [17]. The same reference also studies a class of sequences that are closely related to the Stirling numbers of the 2nd kind. The ρ -Laplace and ρ -Fourier transforms of the Katugampola fractional operators are given in [7].

In the following, we will give some necessary definitions and preliminary results which are used and referred to throughout this paper.

Definition 1.2 ([33]). Let $\alpha > 0$ with $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $a < x < b$. The left- and right-side Riemann–Liouville fractional integrals of order α of a function f are given by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \quad \text{and} \quad J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt$$

respectively, where $\Gamma(\cdot)$ is the Euler’s gamma function defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

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