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An orthogonal projection related to the Riemann zeta-function

Xian-Jin Li

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA

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ABSTRACT

In this paper, a global orthogonal projection is found for which an explicit formula is given. Under the assumption of a density result, this projection denies the existence of zeros of the Riemann zeta-function on the right side of the critical line, but not zeros on the left side.

1. Introduction

In trying to understand the underlying space associated with Connes' trace formula [2], the author decomposed in a previous paper [5] a S-local L^2 space into a direct sum of a Sonine space [4], the space of prolate spheroidal wave functions [7], and a variant of the space of prolate spheroidal wave functions. The difficulty to extend this decomposition to global case is to find computable global orthogonal projections. As an initial step in this direction, the author found a nice global orthogonal projection.

To describe the results, we need to introduce some notations as follows: Let K denote the field of rational numbers. For every place v, we denote by K_v, O_v, and P_v the completion of K at v, the maximal compact subring of K_v, and the unique maximal ideal of O_v, respectively. The ring of adeles and the group of ideles of K are denoted by $\mathbb{A}$ and J_K, respectively. Let $C_K = J_K/K^*$, and J_K^1 the set of ideles $\alpha = (\alpha_v)$ such that $\prod |\alpha_v|_v = 1$.

For every place v of K we denote by $||_v$ the valuation of K normalized so that $||_v$ is the ordinary absolute value if v is real, and $|\pi_v|_v = p^{-1}$ if O_v/P_v contains p elements where $P_v = \pi_v O_v$ (for clarity, we sometimes write p_v instead of p). Let ψ_v be a character on the additive group K_v given as in [8]. In particular, $\psi_\infty(x) = \exp(-2\pi i x)$. Also, $\psi = \prod_v \psi_v$ is a character on $\mathbb{A}$ satisfying $\psi(\alpha) = 1$ for all $\alpha \in k$.

For each place v of K, we select a fixed Haar measure $d\alpha_v$ on the additive group K_v as follows: $d\alpha_v =$ the ordinary Lebesgue measure on the real line if v is real, and $d\alpha_v =$ that measure for which O_v get measure 1 if v is finite. Then a unique Haar measure $d\alpha = \prod_v d\alpha_v$ on $\mathbb{A}$ exists such that

E-mail address: xianjin@byu.edu.

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$$\int_{\mathbb{A}} f(\alpha)\, d\alpha = \prod_v \int_{K_v} f_v(\alpha_v)\, d\alpha_v$$

for every function f of the form $f(\alpha) = \prod_v f_v(\alpha_v) \in L_1(\mathbb{A})$ provided $f_v = 1_{O_v}$ for almost all v, where 1_{O_v} denotes the characteristic function of O_v.

The Fourier transform $\widehat{f}$ of $f \in L^2(\mathbb{A})$ is defined by

$$\widehat{f}(\beta) = \int_{\mathbb{A}} f(\alpha)\psi(-\alpha\beta)\, d\alpha.$$

For $f \in L^2(\mathbb{R})$ we denote

$$(H_\infty f)(x) = \int_{-\infty}^{\infty} f(t)e^{2\pi ixt}dt,$$

and for $f \in L^2(K_v)$ we write

$$H_v f(\beta) = \int_{K_v} f(\alpha)\psi_v(-\alpha\beta)d\alpha$$

for $\beta \in K_v$.

The finite set S of primes is always chosen large enough so that it contains ∞ and all primes up to a certain bound. Put $S' = S - \{\infty\}$.

We denote by $\mathbb{N}^S$ the set of all positive integers n such that $(n, p) = 1$ for all $p \in S'$. Define

$$e^S f_S(x_S) = \sum_{n\in\mathbb{N}^S} f_S(nx_S) \text{ and } (e^S)^{-1} f_S(x_S) = \sum_{n\in\mathbb{N}^S} \mu(n) f_S(nx_S)$$

for every f whenever the series converges, where $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k distinct primes, and $\mu(n) = 0$ if n contains any factor to a power higher than the first.

Let $S_r(\mathbb{A}_S)_0 = \left\{f_\infty \prod_{v\in S'} 1_{O_v} : f_\infty \text{ are even functions in the Schwartz space } S(\mathbb{R})_0\right\}$, and let $\mathcal{H}(C_S)$ be the completion of $E_S e^S\,(S_r(\mathbb{A}_S)_0)$ for the norm $\|\ \|_{\mathcal{H}(C_S)}$ given by

$$\|E_S e^S(f_S)\|^2_{\mathcal{H}(C_S)} = \int_{C_S} |E_S e^S(f_S)(x_S)|^2 d^\times x_S, \tag{1.1}$$

where $E_S(f_S)(x_S) = \sqrt{|x_S|}\sum_{\xi\in O_S^*} f_S(\xi x_S)$ with $O_S^* = \{\xi \in K^* :\ |\xi|_v = 1,\ v \notin S\}$.

We define $\mathcal{H}(X_S)$ to be the Hilbert space that is the completion of $S_r(\mathbb{A}_S)_0$ for the inner product induced by the norm

$$\|f_S\|_{\mathcal{H}(X_S)} = \|E_S e^S(f_S)\|_{\mathcal{H}(C_S)} \tag{1.2}$$

for $f_S \in S_r(\mathbb{A}_S)_0$.

Let

$$P_\Lambda = \{f_S \in \mathcal{H}(X_S) : f_S(x_S) = 0\ |x_S| \geq \Lambda\}.$$

The corresponding orthogonal projection is still denoted by P_Λ.