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Asymptotic spectral analysis in colliding leaky quantum layers

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ABSTRACT

We consider the Schrödinger operator with a complex delta interaction supported by two parallel hypersurfaces in the Euclidean space of any dimension. We analyse spectral properties of the system in the limit when the distance between the hypersurfaces tends to zero. We establish the norm-resolvent convergence to a limiting operator and derive first-order corrections for the corresponding eigenvalues.

1. Introduction

Semiconductor heterostructures have had tremendous impact on science and technology as building blocks for a bottom-up approach to the fabrication of nanoscale devices. A key property of these material systems is the unique versatility in terms of geometrical dimensions and composition and their ability to exhibit quantum effects. Theoretical studies have lead to interesting mathematical problems which involve an interaction of differential geometry, spectral analysis and theory of partial differential equations. In this paper, we rely on the mathematical concept of *leaky quantum graphs* or *waveguides* introduced by Exner and Ichinose in 2001 [15] (see [14] for a survey), where the quantum Hamiltonian is modelled by the Schrödinger operator with a Dirac-measure potential supported on a hypersurface in $\mathbb{R}^d$.

The situations $d = 1, 2, 3$ are of particular interest in the context of mesoscopic physics of nanostructures, where they are sometimes referred to as *quantum dots*, *wires* or *layers*, respectively. We adopt the last terminology to emphasise the geometric complexity of the problem, but any value $d \geq 1$ is allowed in this paper. Using the Dirac-measure interaction instead of a regular potential to describe a quantum particle in

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a nanostructure is a simplification in the sense that the former vanishes outside the hypersurface. At the same time, it is a more realistic model than considering the particle confined to a tubular neighbourhood of the hypersurface by means of Dirichlet boundary conditions (see [12,7,31,33,26] for this type of models in the case $d = 3$), because it takes into account *tunnelling*, property which is observed and measured in realistic heterostructures (see, *e.g.*, [6] and [9]).

The objective of this paper is to quantify the effect of tunnelling by considering coalescing heterostructures modelled by Dirac-measure potentials imposed on two parallel hypersurfaces separated by a distance ε and studying spectral properties in the limit as ε tends to zero. Spectral asymptotics of systems with leaky quantum waveguides have been analysed in various contexts and dimensions recently (see, *e.g.*, [3–5, 11,17,23,32]). The geometric setting introduced in this paper is new and interesting both physically and mathematically. In fact, to establish the eigenvalue asymptotics as $\varepsilon \to 0$, we need to combine diverse methods of Riemannian geometry, spectral analysis and theory of partial differential equations.

Motivated by a growing interest in non-self-adjoint operators in recent years (*cf.* the review article [29] and the book chapter [28] and references therein), in this paper we proceed in a great generality by allowing *complex* couplings on the colliding hypersurfaces. In quantum mechanics, non-self-adjoint operators are traditionally relevant as effective models of open systems and, more recently, as an unconventional representation of physical observables. Schrödinger operators with complex delta interactions are specifically used in Bose–Einstein condensates, where the imaginary part of the complex coupling models the injection and removal of particles (see [8] and [10]).

Let us now specify the mathematical model of this paper and present our main results. Let Ω be a bounded smooth open set in $\mathbb{R}^d$ with $d \geq 1$ and let us denote by $\Sigma_0 := \partial\Omega$ the boundary of Ω. For all sufficiently small positive ε, we consider parallel hypersurfaces

$$\Sigma_{\pm\varepsilon} := \{q \pm \varepsilon n(q) : q \in \Sigma_0\}\,, \tag{1.1}$$

where $n : \Sigma_0 \to \mathbb{R}^d$ denotes the outer unit normal to Ω. Finally, given two constants $\alpha_\pm \in \mathbb{C}$, we consider the operator in $L^2(\mathbb{R}^d)$ represented by the formal expression

$$H_\varepsilon := -\Delta + \alpha_+\,\delta_{\Sigma_{+\varepsilon}} + \alpha_-\,\delta_{\Sigma_{-\varepsilon}}\,, \tag{1.2}$$

where δ_Σ denotes the Dirac delta function supported by a hypersurface $\Sigma \subset \mathbb{R}^d$. The purpose of this paper is to study spectral properties of H_ε in the limit when $\varepsilon \to 0$.

First of all, it is natural to expect that the limiting operator is given by

$$H_0 := -\Delta + (\alpha_+ + \alpha_-)\,\delta_{\Sigma_0}\,. \tag{1.3}$$

In this paper, we show that the convergence holds in the norm-resolvent sense.

Theorem 1.1. *For any $z \in \rho(H_0)$, there exists a positive constant ε_0 such that, for all $\varepsilon < \varepsilon_0$, we have $z \in \rho(H_\varepsilon)$ and*

$$\left\|(H_\varepsilon - z)^{-1} - (H_0 - z)^{-1}\right\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(\varepsilon) \qquad as \qquad \varepsilon \to 0\,. \tag{1.4}$$

As a consequence of Theorem 1.1, we obtain a convergence of the spectrum of H_ε to the spectrum of H_0 as $\varepsilon \to 0$. In particular, discrete eigenvalues change continuously with ε (*cf.* [21, Sec. IV.3.5]). By a discrete eigenvalue λ_ε of H_ε we mean an isolated eigenvalue of finite algebraic multiplicity such that the range of $H_\varepsilon - \lambda_\varepsilon$ is closed. We remark that H_0 may or may not possess discrete eigenvalues, depending on the values of the coupling constants $\alpha_\pm$ and geometry of Σ_0; in particular, they always exist in the self-adjoint case if the constants are negative and sufficiently large. Since the interaction in (1.2) is compactly supported in $\mathbb{R}^d$, it