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Factorization in Lorentz spaces, with an application to centralizers ☆

Félix Cabello Sánchez [1]

Departamento de Matemáticas, UEx, 06071 Badajoz, Spain

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ABSTRACT

We give an explicit formula to factorize functions in the Lorentz spaces $L(p,q)$. Some applications to centralizers, twisted sums and interpolation theory are included.

1. Introduction

1.1. Purpose

The purpose of this paper is rather modest. We aim to factorize functions as a product whose factors have "small Lorentz norms".

To motivate the problem let us consider first Lozanovsky factorizations (see [21, p. 646] for the historical background). Suppose X is a Banach function space with Köthe dual X'. An important result of Lozanovsky states that every $f \in L_1$ can be written as $f = gh$, where $g \in X$, $h \in X'$ and $\|g\|_X \|h\|_{X'} \leq (1+\varepsilon)\|f\|_1$. Thus, for instance, when $X = L_p$ with $1 < p < \infty$ one has $X' = L_{p'}$ where $1/p + 1/p' = 1$ and every $f \in L_1$ can be factorized as $f = u|f|^{1/p}|f|^{1/p'}$ with $\|u|f|^{1/p}\|_p \||f|^{1/p'}\|_{p'} = \|f\|_1$. Here, $f = u|f|$ is the "polar decomposition".

When $X = L(p,q)$ is a Lorentz space, (with $p, q > 1$) its Köthe dual is (isomorphic, but not isometric to) $L(p', q')$ and so every integrable f can be written as $f = gh$, with $\|g\|_{p,q}\|h\|_{p',q'} \leq M\|f\|_1$ for some constant M that may depend on p and q, but not on f.

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E-mail address: fcabello@unex.es.

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Surprisingly enough in this case no explicit formula seems to be available. Even if the meaning of "explicit" may be a matter of perspective, we believe that no factorization explicit enough to write down the centralizers associated to Lorentz spaces was previously known; see Section 3.

1.2. Summary

The functionals used to measure size in Lorentz spaces are not generally norms, but merely quasinorms. For this reason, among others, one cannot expect to get estimates with constant close to one and so we prefer to treat Lorentz spaces as quasi-Banach spaces from the start and we consider the whole family $L(p, q)$ for $0 < p, q \leq \infty$.

The main statement of the paper is the following.

Theorem. *Suppose* $(p_0, q_0)^{-1} + (p_1, q_1)^{-1} = (p, q)^{-1}$. *Then there is a constant* M *depending only on* (p_0, q_0) *and* (p_1, q_1) *such that:*

(a) *If* $f_i \in L(p_i, q_i)$ *for* $i = 0, 1$, *then* $f_0 f_1 \in L(p, q)$ *and* $\|f_0 f_1\|_{p,q} \leq M\|f_0\|_{p_0,q_0}\|f_1\|_{p_1,q_1}$.
(b) *If* $f \in L(p, q)$, *then there are* $f_i \in L(p_i, q_i)$ *for* $i = 0, 1$ *such that* $f = f_0 f_1$ *and* $\|f_0\|_{p_0,q_0}\|f_1\|_{p_1,q_1} \leq M\|f\|_{p,q}$.
(c) *If* $q < \infty$ *and* $f \geq 0$, *one can take* $f_i = f^{qq_i^{-1}} r_f^{qp^{-1}q_i^{-1} - p_i^{-1}}$ *for* $i = 0, 1$ *in* (b), *where* r_f *is the rank function of* f. *For* $q = \infty$ *and* $f \geq 0$ *one may take* $f_i = f^{pp_i^{-1}}$ *in* (b).

The Hölder type inequality in (a) is a classical result that appears, for instance, in [12, Theorem 4.5] or [21, Lemma 4]. The constant M arises from the use of the Hardy operator in the proof. Actually, we can state (a) and (b) together by saying that $L(p, q) = L(p_0, q_0)L(p_1, q_1)$ provided the parameters satisfy the hypothesis of the Theorem; see the beginning of Section 3.1. This fact is roughly equivalent to Calderón formula

$$[L(p_0, q_0), L(p_1, q_1)]_\theta = L(p, q), \quad \text{where } (p, q)^{-1} = (1 - \theta)(p_0, q_0)^{-1} + \theta(p_1, q_1)^{-1}, \theta \in [0, 1],$$

for the complex interpolation method. Thus, the fact that each $f \in L(p, q)$ can be factorized as $f = f_0 f_1$ with $\|f_0\|_{p_0,q_0}\|f_1\|_{p_1,q_1} \leq M\|f\|_{p,q}$ for some constant M independent on f is well known. This is implicit in [7,12] and stated and proved in [21].

So, the real contribution of the paper is the explicit factorization that appears in (c), whose proof occupies the entire Section 2 and the identification of the centralizers associated to the Lorentz spaces which is carried out in Section 3. Those readers that are primarily interested in centralizers and their connections with twisted sums and interpolation theory may skip Section 2 and take a look to Section 4 before going into Section 3.

1.3. Notations, conventions

The first part of this note is more or less self-contained. Our main sources on Lorentz spaces have been the classical paper by Hunt [12] and Dilworth's article in the Handbook [10]. As for rearrangements and the rank function we have followed Ryff's nice note [26]. A good reference for centralizers is Kalton's memoir [15] and the connections between centralizers and complex interpolation theory can be seen in [19].

In this Section we gather some definitions and basic facts about rearrangements and Lorentz spaces we shall use along the proofs. We consider the half-line $\mathbb{R}_+ = (0, \infty)$ equipped with Lebesgue measure m and we denote by L_0 the space of all (complex-valued) measurable functions on $\mathbb{R}_+$. As usual we identify functions that agree almost everywhere. To each $f \in L_0$ we attach the following items: