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Heat content for convolution semigroups

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A B S T R A C T

Let $\mathbf{X} = \{X_t\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ and Ω be an open subset of $\mathbb{R}^d$ with finite Lebesgue measure. In this article we consider the quantity $H(t) = \int_\Omega \mathbb{P}_x(X_t \in \Omega^c)\,\mathrm{d}x$ related to $\mathbf{X}$ which is called the heat content. We study its asymptotic behaviour as t goes to zero for isotropic Lévy processes under some mild assumptions on the characteristic exponent. We also treat the class of Lévy processes with finite variation in full generality.

1. Introduction

Let $\mathbf{X} = \{X_t\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ with the distribution $\mathbb{P}$ such that $X_0 = 0$. We denote by $p_t(\mathrm{d}x)$ the distribution of the random variable X_t and we use the standard notation $\mathbb{P}_x$ for the distribution related to the process $\mathbf{X}$ started at $x \in \mathbb{R}^d$. The characteristic exponent $\psi(x)$, $x \in \mathbb{R}^d$, of the process $\mathbf{X}$ is given by the formula

$$\psi(x) = \langle x, Ax\rangle - i\langle x, \gamma\rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,y\rangle} - 1 - i\langle x, y\rangle \mathbf{1}_{\{\|y\|\leq 1\}}\right) \nu(\mathrm{d}y), \tag{1}$$

where A is a symmetric non-negative definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, that is

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu(\mathrm{d}y) < \infty. \tag{2}$$

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Let Ω and Ω_0 be two non-empty subsets of $\mathbb{R}^d$ such that Ω is open and its Lebesgue measure $|\Omega|$ is finite. We consider the following quantity associated with the process $\mathbf{X}$,

$$H_{\Omega,\Omega_0}(t) = \int_{\Omega} \mathbb{P}_x(X_t \in \Omega_0)\,\mathrm{d}x = \int_{\Omega}\int_{\Omega_0 - x} p_t(\mathrm{d}y)\mathrm{d}x$$

and we use the notation

$$H_\Omega(t) = H_{\Omega,\Omega}(t) \quad \text{and} \quad H(t) = H_{\Omega,\Omega^c}(t). \tag{3}$$

The main goal of the present article is to study the asymptotic behaviour of $H_\Omega(t)$ as t goes to zero. We observe that

$$H_\Omega(t) = |\Omega| - H(t),$$

and thus it suffices to work with the function $H(t)$. The function $u(t,x) = \int_{\Omega - x} p_t(\mathrm{d}y)$ is the weak solution of the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= -\mathcal{L}\, u(t,x), \quad t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) &= \mathbf{1}_\Omega(x), \end{aligned}$$

where $\mathcal{L}$ is the infinitesimal generator of the process $\mathbf{X}$, see [17, Section 31]. Therefore, $H_\Omega(t)$ can be interpreted as the amount of *heat* in Ω if its initial temperature is one whereas the initial temperature of Ω^c is zero. In paper [19], the author calls the quantity $H_\Omega(t)$ *heat content* and we will use the same terminology. There are a lot of articles where bounds and asymptotic behaviour of the heat content related to Brownian motion, either on $\mathbb{R}^d$ or on compact manifolds, were studied, see [19,21,22,20,18,23]. Recently Acuña Valverde [2] investigated the heat content for isotropic stable processes in $\mathbb{R}^d$, see also [1] and [3]. In this paper we study the small time behaviour of the heat content associated with rather general Lévy processes in $\mathbb{R}^d$.

Before we state our results we recall the notion of perimeter. Following [4, Section 3.3], for any measurable set[3] $\Omega \subset \mathbb{R}^d$ we define its perimeter $\mathrm{Per}(\Omega)$ as

$$\mathrm{Per}(\Omega) = \sup\left\{ \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) \mathrm{div}\,\phi(x)\,\mathrm{d}x : \ \phi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d),\ \|\phi\|_\infty \le 1 \right\}. \tag{4}$$

We say that Ω is of finite perimeter if $\mathrm{Per}(\Omega) < \infty$. It was shown in [13–15] that if Ω is an open set in $\mathbb{R}^d$ with finite Lebesgue measure and of finite perimeter then

$$\mathrm{Per}(\Omega) = \pi^{1/2} \lim_{t \to 0} t^{-1/2} \int_{\Omega}\int_{\Omega^c} p_t^{(2)}(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

where

$$p_t^{(2)}(x,y) = (4\pi t)^{-d/2} e^{-\|x-y\|^2/4t}$$

is the transition density of the Brownian motion B_t in $\mathbb{R}^d$. We also notice that for a non-empty and open set Ω, $\mathrm{Per}(\Omega) > 0$.

[3] All sets in the paper are assumed to be Lebesgue measurable.