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Global existence result for chemotaxis Navier–Stokes equations in the critical Besov spaces

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We consider the chemotaxis Navier–Stokes equations which are the Keller–Segel model coupled to the incompressible Navier–Stokes equations in dimension three. We prove the global existence of the solution in the critical Besov spaces with small initial data only, which is a new result and improves the previous result.

1. Introduction

We study the chemotaxis Navier–Stokes equations which describe the dynamics of oxygen, moving bacteria and viscous incompressible fluids. Chemotaxis is the movement of living cells under the effects of chemical attractants. Aerobic bacteria often live in thin fluid layers near solid-air-water contact lines, in which bacteria move towards higher oxygen concentration region according to the chemotaxis and at the same time the flow of fluid is under the influence of gravitational force generated by bacteria. Both the oxygen concentration and bacteria density are transported by the fluid and diffuse through the fluid. We consider such a model, which was proposed by Tuval et al. [9], to describe the dynamics of swimming bacteria, in the following form

$$\begin{aligned} \partial_t n + (u\cdot\nabla)n - \Delta n &= -\nabla\cdot(\chi(c)n\nabla c) \\ \partial_t c + (u\cdot\nabla)c - \Delta c &= -k(c)n \\ \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p &= -n\nabla\phi, \quad \nabla\cdot u = 0 \end{aligned} \tag{1}$$

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in $\mathbb{R}^3$ with initial data (n_0, c_0, u_0). The state variables c, n, u and p denote the chemical concentration, cell density, fluid velocity field and pressure of the fluid, respectively. The function χ is called chemotactic sensitivity and k denotes the consumption rate of chemicals by the bacteria. The time independent function $\phi = \phi(x)$ denotes the gravitational potential.

These equations have been extensively studied by many authors due to the significance of the biological background. The local existence of a weak solution to the system (1) in a bounded domain in $\mathbb{R}^d, d = 2, 3$ was obtained in [8]. The existence of unique global classical solution for bounded convex domain was proved in [10] for the case $d = 2$. The global in time existence of the smooth solutions was proved when the initial data are close to constant steady states in $\mathbb{R}^3$ and χ, k satisfy certain conditions in [6]. The local wellposedness and blow up criterion of smooth solutions of (1) in the framework of H^m with $m \geq 3$ in $\mathbb{R}^d, d = 2, 3$ were proved in [3] and [4]. Under the condition $\chi(c) = 1$ and $k(c) = c$, the local wellposedness and Beale–Kato–Majda type blow up criterion with logarithmic inequality in the framework of nonhomogeneous Besov space $B^s_{p,r}$ with $1 < p < \infty$, $1 \leq r \leq \infty$ and $s > d/p + 1$ for $d = 2, 3$ was obtained by Zhang in [11]. In the critical Besov spaces, the local in time existence of the solution for large initial data and the global in time existence of the solution for small initial data plus some smallness condition on the gravitational potential were obtained in [5]. However, whether solutions of (1) with large initial data exist globally or blow up appears to remain an open problem. We refer to [3,4,11] and references therein for recent results on other variants of the model (1).

In this paper, we prove the global existence result for small initial data only. This improves the result in [5], where authors used contraction mapping principle. We assume $\chi(c) = 1$ and $k(c) \leq Ac$ for some constant A as in [5]. We say that (n, c, u) is a mild solution of the chemotaxis Navier–Stokes equations on $[0, T]$ if (n, c, u) solves for $0 \leq t \leq T$ the integral equations

$$
\begin{aligned}
n(t,x) &= S(t)n_0 - \int_0^t S(t-\tau)\Big(u\cdot\nabla n + \nabla\cdot(n\nabla c)\Big)d\tau, \\
c(t,x) &= S(t)c_0 - \int_0^t S(t-\tau)\Big(u\cdot\nabla c + k(c)n\Big)d\tau, \\
u(t,x) &= S(t)u_0 - \int_0^t S(t-\tau)\mathbb{P}\Big(\nabla\cdot(u\otimes u) + n\nabla\phi\Big)d\tau,
\end{aligned}
\tag{2}
$$

where $S(t) = e^{t\Delta}$ denotes the heat semigroup and $\mathbb{P}$ is Helmholtz projection operator.

The following theorem is our main result:

Theorem 1. *Let $\phi \in \dot{B}^{3/p}_{p,1}$, $1 \leq p < 3$ and the initial data is given by $(n_0, c_0, u_0) \in \dot{B}^{-2+3/p}_{p,1} \times \dot{B}^{3/p}_{p,1} \times \dot{B}^{-1+3/p}_{p,1}$. Then there exists a constant ξ such that if $\|n_0\|_{\dot{B}^{-2+3/p}_{p,1}} + \|c_0\|_{\dot{B}^{3/p}_{p,1}} + \|u_0\|_{\dot{B}^{-1+3/p}_{p,1}} < \xi$, then the equations (1) have a unique mild solution (n, c, u) in*

$$
\Big(C(\mathbb{R}^+; \dot{B}^{-2+3/p}_{p,1}) \times C(\mathbb{R}^+; \dot{B}^{3/p}_{p,1}) \times C(\mathbb{R}^+; \dot{B}^{-1+3/p}_{p,1})\Big) \cap \Big(\widetilde{\mathscr{X}}^1 \times \widetilde{\mathscr{X}}^2 \times \widetilde{\mathscr{X}}^3\Big)
$$

where the spaces $\widetilde{\mathscr{X}}^1, \widetilde{\mathscr{X}}^2$ and $\widetilde{\mathscr{X}}^3$ are defined by

$$
\begin{aligned}
\widetilde{\mathscr{X}}^1 &= \widetilde{L^\infty}_{[0,\infty]}(\dot{B}^{-2+3/p}_{p,1}) \cap \widetilde{L^1}_{[0,\infty]}(\dot{B}^{3/p}_{p,1}), \\
\widetilde{\mathscr{X}}^2 &= \widetilde{L^\infty}_{[0,\infty]}(\dot{B}^{3/p}_{p,1}) \cap \widetilde{L^1}_{[0,\infty]}(\dot{B}^{2+3/p}_{p,1}), \\
\widetilde{\mathscr{X}}^3 &= \widetilde{L^\infty}_{[0,\infty]}(\dot{B}^{-1+3/p}_{p,1}) \cap \widetilde{L^1}_{[0,\infty]}(\dot{B}^{1+3/p}_{p,1})
\end{aligned}
$$