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Differential inclusions with state-dependent impulses on the half-line: New Fréchet space of functions and structure of solution sets

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ABSTRACT

An R_δ-structure of solution sets to some classes of impulsive differential inclusions with variable impulse times on the half-line is shown. Sufficient conditions on barriers are discussed in detail. They imply two different techniques: the inverse limit method and the one based on a definition of R_δ-sets and used in a new suitable Fréchet function space.

1. Introduction

The goal of the paper is to study a topological structure of the set of solutions to the following impulsive multivalued problem

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{for a.a. } t \in [0,\infty),\ t \neq \tau_j(y(t)),\ j \in \mathbb{N}, \\ y(0) = x_0, & \\ y(t^+) = y(t) + I_j(y(t)), & \text{for } t = \tau_j(y(t)),\ j \in \mathbb{N}, \end{cases} \tag{1}$$

where, as one can see, there are infinitely many impulses at variable times (state-dependent impulses).

The exploration of the area of impulsive differential equations has a quite long history. It started by [18] in 1960 and has been developed in many directions but, in fact, problems with pulse action are met already in [7]. Moreover, a lot of processes in biology, economics, medicine, physics and other fields have been modeled by impulsive equations or inclusions (see, e.g., [2,6,11,15,19] and references therein).

For problems with fixed moments of impulses (i.e., $\tau_j \equiv \text{const}$) several methods from continuous problems can be simply adopted (see [12] and references therein). Problems with state-dependent impulses are much

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more complicated, especially the qualitative ones. One of the obstacles is related with a topology in a suitable function space, where all solutions could be embedded. Main techniques to examine a topological structure of the solution set require a Banach space regularity (an absolute neighborhood retract regularity, at least), see e.g. [9,11]. Therefore a B-topology proposed by Akhmet in [2] is insufficient. Recently, in [16] a nice function Banach space $CJ_m([0,a])$ was defined and, under suitable assumptions, all solutions on a compact interval $[a,b]$ were interpreted as elements of this space. This allowed the authors to prove that the solution set of the Cauchy multivalued problem is an R_δ-set. Note that the R_δ structure is very important in many applications and has been studied since the well known Aronszajn theorem was published in 1942, see [4]. Roughly speaking, R_δ-sets are compact and acyclic, that is, they have the same homology groups as singletons.

Impulsive problems on the half-line $[0,\infty)$ bring another difficulty, namely, a lifetime of some trajectories can be finite even if each barrier (a hyperspace of impacts) is met exactly once. Therefore in the paper we devote some space to analysis of this phenomenon.

The existence of solutions for differential problems on noncompact intervals can be obtained by various techniques while the inverse limit method proposed in [13] and [3] is very useful for results on a topological structure of solution sets for such problems (see, e.g., [10] where this method is applied for problems with impulses in fixed moments). Unfortunately, this fruitful method is of limited use for impulsive problems with state-dependent impulses. Nevertheless we describe in the paper the situation this technique is applicable. In more general cases we are forced to look for more sophisticated tools.

The paper is organized as follows. Section 2 contains some basic notions and results we use in main parts of the paper, and we provide a new Fréchet space such that all solutions to problem (1) can be interpreted as its elements. Since every Fréchet space is an ANR, it becomes a good environment to examine a topological structure of the solution set. Section 3 is divided into three subsections. In the first one we give and analyze several assumptions we will need to prove main results. Especially we comment conditions on barriers guaranteeing that solutions meet every barrier exactly once and their lifetime is infinite. In subsection 3.2 the inverse limit technique is applied to prove the first theorem on an R_δ structure. Here the results from [16] are used to build an inverse system of sets with an inverse limit being the solution set for (1). The second main result, with a quite long proof, is contained in subsection 3.3. One illustrative example is also given. The paper finishes with some concluding remarks because it was not the goal of the paper to present results in the highest generality with more complicated formulation and proofs. We find the construction of a suitable function space $\mathcal{PC}_\infty$ and presentation of the idea of proof more valuable and transparent.

2. Preliminaries

Let X, Y be topological spaces. In the whole paper by a *multivalued map* from X to Y we mean a function, denoted by $F : X \multimap Y$ with domain X and values being nonempty subsets of Y. A multivalued map $F : X \multimap Y$ is said to be *upper semicontinuous* (for short u.s.c.) if $F^{-1}(V) = \{x \in X \mid F(x) \subset V\}$ is an open subset of X for every open $V \subseteq Y$.

Recall that a subset $A \subset X$ is called a *retract* of X if there exists a continuous function $r : X \to A$ such that $r(x) = x$ for every $x \in A$. A is a *neighborhood retract* of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is a retract of U. If we have two spaces X, Y, then every homeomorphism $h : X \to Y$ such that $h(X)$ is a closed subset of Y is called an *embedding*. We say that X is an *absolute retract* (*absolute neighborhood retract*) if for any space Y and for any embedding $h : X \to Y$ the set $h(X)$ is a retract of Y (resp. a neighborhood retract of Y). We write $X \in AR$ (resp. $X \in ANR$).

A compact nonempty space is called an R_δ*-set* provided there exists a decreasing sequence $\{A_n\}$ of compact absolute retracts such that $A = \bigcap_{n\geq 1} A_n$.

We will use the following characterization of R_δ-sets.