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Local isometric immersions of pseudospherical surfaces described by evolution equations in conservation law form

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ABSTRACT

For the pseudospherical surfaces described by a class of second order evolution equations, of the form $z_t = A(x,t,z)\,z_2 + B(x,t,z,z_1)$, we consider the problem of local isometric immersion into the 3-dimensional Euclidean space $\mathbf{E}^3$ with a second fundamental form depending on finite-order jets of solutions z of the considered equations. We also provide an extension of our analysis to the case of k-th order evolution equations in conservation law form. Examples of equations admitting such local isometric immersions, are provided by equations like Burgers, Murray, Svinolupov–Sokolov, Kuramoto–Sivashinsky, Sawada–Kotera, Kaup–Kupershmidt, as well as hierarchies of evolution equations in conservation law form like Burgers, mKdV and KdV.

1. Introduction

The best known example of a differential equation which describes pseudospherical surfaces (PS equation) is certainly the celebrated sine-Gordon equation $z_{xt} = sin(z)$, which in the early studies of classical theory of surfaces in the 3-dimensional Euclidean space $\mathbf{E}^3$ was discovered to be equivalent to the Gauss–Codazzi equations for pseudospherical surfaces in terms of Darboux asymptotic coordinates [35]. This equation is also the first well known example of an equation integrable by means of geometric techniques originating from the classical theory of surface transformation, and first applied to it by Bäcklund and Bianchi.

More in general PS equations arise ubiquitously in the description of many nonlinear physical phenomena, as well as in many problems of pure and applied mathematics. In particular, in the theory of solitons and integrable systems, the interest on the general study of PS equations began with the early observation [37] that "all the soliton equations in $1+1$ dimensions that can be solved by the AKNS 2×2 inverse scattering

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method (for example, the sine-Gordon, KdV or modified KdV equations) ... describe pseudospherical surfaces". This motivated a general study of these equations, initiated with the fundamental paper [14] by S.S. Chern and K. Tenenblat, which lead to important geometric interpretations of Bäcklund transformations, conservation laws, non-local symmetries and correspondences between solutions of different equations, for equations of this class [4,24,30,34]. The results of this study, together with the considerable effort addressed over the past few decades to the possible applications of inverse scattering method [1–3,17], gave a significant contribution to the discovery of new integrable equations. For instance, Belinski–Zakharov system in General Relativity [5], the Jackiw–Teitelboim two-dimensional gravity model [18,33], the nonlinear Schrödinger type systems [13,15,16], the Rabelo's cubic equation [4,25,26,36], the Camassa–Holm, Degasperis–Procesi, Kaup–Kupershmidt and Sawada–Kotera equations [6,9,29,31,32,34] are some important examples of equations describing pseudospherical surfaces and integrable by inverse scattering method. All these facts prove the relevance of these equations and justify the general interest in their systematic study. The reader is referred to [12,14,21,24–27] for an account of the early studies on the geometry of PS equations, and to [34] for a general review of existing literature on the subject. In particular, for an account of the various contributions to the problem of classifying PS equations, the reader is referred to [8–11,14,15,19,25–28, 30].

From a geometric point of view, every PS equation $\mathcal{E}$ satisfies the following remarkable property: to any generic solution (see below) $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ of $\mathcal{E}$ it is associated a Riemannian metric on the domain U with Gaussian curvature $K = -1$. Indeed, by definition a differential equation $\mathcal{E}$ for a real function $z = z(x,t)$ is a PS equation if it is equivalent to the structure equations $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$ of a 2-dimensional Riemannian manifold whose Gaussian curvature $K = -1$, with 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ satisfying the independence condition $\omega_1 \wedge \omega_2 \neq 0$ and such that f_{ij} are smooth functions of x, t, z and a finite number of derivatives of z with respect to x and t. For any solution $z = z(x,t)$, we will denote by $(\omega_1 \wedge \omega_2)[z]$ the restriction of $\omega_1 \wedge \omega_2$ to $z(x,t)$ and its partial derivatives. We notice that the independence condition $\omega_1 \wedge \omega_2 \neq 0$ does not guarantee the property that, for any solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ the 2-form $(\omega_1 \wedge \omega_2)[z]$ is everywhere nonzero on U. We will call *generic* a solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $(\omega_1 \wedge \omega_2)[z]$ is almost everywhere nonzero on U, i.e., it is everywhere nonzero except for a subset of U of measure zero. Thus, for any generic solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ of a PS equation $\mathcal{E}$, the restriction $I[z]$ of $I = \omega_1^2 + \omega_2^2$ to $z = z(x,t)$ and its partial derivatives defines almost everywhere a Riemannian metric $I[z]$ on the domain U with Gaussian curvature $K = -1$. It is in this sense that one can say that a PS equation describes, or parametrizes, non-immersed pseudospherical surfaces.

For instance, one may easily check that sine-Gordon equation $z_{xt} = sin(z)$ is equivalent to the above structure equations for the following system of 1-forms

$$\begin{aligned} \omega_1 &= \tfrac{1}{\eta} sin\,(z)\;dt, \\ \omega_2 &= \eta\, dx + \tfrac{1}{\eta} cos\,(z)\;dt, \\ \omega_3 &= z_x\, dx, \end{aligned} \tag{1.1}$$

with $\eta \in \mathbb{R} - \{0\}$. In this case one has that $I = \omega_1^2 + \omega_2^2 = \frac{1}{\eta^2}dt^2 + 2\,cos(z)\,dx\,dt + \eta^2 dx^2$. Notice that, with respect to the system (1.1), sine-Gordon equation admits non-generic solutions. For instance, $z = k\pi$, $k \in \mathbb{Z}$, is a non-generic solution of sine-Gordon equation.

Taken separately, any pseudospherical surface described by a PS equation $\mathcal{E}$ admits a local isometric immersion into $\mathbf{E}^3$: it is due to the classical result that a pseudospherical surface always admits a local isometric immersion into $\mathbf{E}^3$. Hence, in view of the Bonnet theorem, to any generic solution z of $\mathcal{E}$ it is associated a pair $(I[z], II[z])$ of first and second fundamental forms, which solves the Gauss–Codazzi equations and describes a local isometric immersion into $\mathbf{E}^3$ of an associated pseudospherical surface. However, the dependence of $(I[z], II[z])$ on z may be quite complicate and in general it is not guaranteed the existence of