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Strong Morita equivalence of operator spaces

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Dedicated to the memory of Uffe Valentin Haagerup

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A B S T R A C T

We introduce and examine the notions of strong Δ-equivalence and strong TRO equivalence for operator spaces. We show that they behave in an analogous way to how strong Morita equivalence does for the category of C*-algebras. In particular, we prove that strong Δ-equivalence coincides with stable isomorphism under the expected countability hypothesis, and that strongly TRO equivalent operator spaces admit a correspondence between particular representations. Furthermore we show that strongly Δ-equivalent operator spaces have stably isomorphic second duals and strongly Δ-equivalent TRO envelopes. In the case of unital operator spaces, strong Δ-equivalence implies stable isomorphism of the C*-envelopes.

1. Introduction

In the 1950's Morita [28] introduced a notion of functorial equivalence for rings. Morita's seminal work was popularized later by Bass [2], and consists mainly of the Morita Theorems I, II, and III; see also [21]. Motivated by the approach of Mackey [25–27] on representations of locally compact groups, Rieffel [31,32] brought the analogues of Morita Theorems into the field of non-commutative geometry. To this end Rieffel introduced a version of Morita equivalence for C*-algebras that implies stronger results. Brown, Green and Rieffel [8,9] introduced later what is sometimes called Morita Theorem IV: in the σ-unital case, strong Morita equivalence coincides with stable isomorphism. The reader is also directed to the survey [33] by Rieffel. The important aspect of strong Morita equivalence and stable isomorphism is the match of the intrinsic structure of C*-algebras that they induce. From one point of view, strong Morita equivalence may be viewed as a generalised unitary equivalence (compare with equation (1.2) that follows).

Their central role in representation theory has been a source of inspiration in the last 20 years for achieving Morita Theorems for a wider range of classes in operator theory. The breakthrough in this direction came

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with the work of Blecher, Muhly and Paulsen [7] on operator algebras for Morita Theorems I and IV. Later Blecher [4] added the relative Morita II and III parts. Extensions to dual operator algebras were given by Blecher and Kashyap [5] and Kashyap [24], for which the first three Morita Theorems were proven. These works rely on the duality flavour of Morita equivalence, i.e. that the algebras X and Y can be decomposed into stabilized tensor products

$$X \simeq M \otimes_Y N \quad \text{and} \quad Y \simeq N \otimes_X M \tag{1.1}$$

of two appropriate bimodules M and N. The notion of tensor product (which varies each time) is used as a generalised multiplication rule.

Nevertheless, strong Morita equivalence in the case of C*-algebras requires that M is a *ternary ring of operators (TRO)*, i.e. $MM^*M \subseteq M$, and implies that $N = M^*$; see for example [34, Section 2]. Then a concrete interpretation hints that a second Morita theory for dual operator algebras is possible by defining X to be *equivalent* to Y when there are completely isometric normal representations ϕ and ψ, and a TRO M such that

$$\phi(X) = [M\psi(Y)M^*]^{-w^*} \quad \text{and} \quad \psi(Y) = [M^*\phi(X)M]^{-w^*}. \tag{1.2}$$

Considering this as the starting point, an alternative approach for dual operator algebras was developed. The notion of Δ-equivalence was introduced by the first author in [12], and the first three Morita Theorems were proven for a certain category of modules over the algebras. The appropriate Morita Theorem IV in this setting was later given by the first author and Paulsen [17]. A further generalisation to the broader class of dual operator spaces was achieved by the first author with Paulsen and Todorov [18].

Both extensions of Morita theory (versions (1.1) and (1.2)) have advantages and a number of applications, e.g. [7, Chapter 8], [5, Examples, pp. 2405–2406] and [18, Section 3]. As they both fit in the wider scheme of functorial equivalence they show resemblances and differences. Relation (1.2) implies relation (1.1) as indicated by Blecher and Kashyap [5, Acknowledgements]. The first author [13] has shown that relation (1.2) is strictly stronger than relation (1.1) for dual operator algebras. This is because relation (1.2) implies all four Morita Theorems, whereas relation (1.1) does not imply in general a Morita IV Theorem. In particular two nest algebras are equivalent in the (1.1) version of [5,24] if and only if the nests are isomorphic, whereas they are equivalent in the (1.1) version of [13] if and only if the isomorphism of the nests extends to an isomorphism of the von Neumann algebras they generate; see the work of the first author [15].

The work on dual operator algebras was recently carried over to operator algebras by the first author [16]. The appropriate Δ-equivalence resembles to the (1.2) relation where the closure is taken in the norm topology. It is shown in [16] that this Morita context is strictly stronger than that of [7], and in addition it satisfies the fourth element.

In the current paper we wish to move forward to the category of operator spaces. By following the Morita context of (1.2) we say that two operator spaces X and Y are *strongly Δ-equivalent* if there are completely isometric representations ϕ and ψ, and two TRO's M_1 and M_2 such that

$$\phi(X) = [M_2\psi(Y)M_1^*]^{-\|\cdot\|} \quad \text{and} \quad \psi(Y) = [M_2^*\phi(X)M_1]^{-\|\cdot\|}, \tag{1.3}$$

(Definition 3.9). Our present aim is to research relations with the aforementioned equivalence relations. We focus on the first and the fourth part of the suggested Morita theory and applications. All results indicate that the (1.3) version has a canonical behaviour and blends well with previous Morita contexts. The other parts of the Morita theory are to pursue elsewhere, as further focus is required for the analysis of the appropriate class of representations.

The first part is devoted in showing that strong Δ-equivalence is indeed an equivalence relation (Theorem 3.11). To this end we also use a concrete version, that of strong TRO equivalence (Subsection 3.1).