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Local smoothing for the quantum Liouville equation

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ABSTRACT

We analyze in this work the regularity properties of the density operator solution to the quantum Liouville equation. As was recently done for the Strichartz inequalities, we extend to systems of orthonormal functions the local smoothing estimates satisfied by the solutions to the Schrödinger equation. We show in particular that the local density associated to the solution to the free, linear, quantum Liouville equation admits locally fractional derivatives of given order provided the data belong to some Schatten spaces.

1. Introduction

The quantum Liouville equation (also referred to as the von Neumann equation) describes the evolution of a (possibly infinite) statistical ensemble of quantum particles. The main object of interest is the *density operator*, denoted by u in the sequel, which is in general a trace class, self-adjoint and nonnegative operator on some Hilbert space (here $L^2(\mathbb{R}^d)$, where $d \geq 1$ is dimension). With physical constants set to one, the free, linear Liouville equation reads

$$\begin{cases} i\partial_t u = \big[-\Delta, u\big] + \varrho, \\ u(t=0) = u_0, \end{cases} \tag{1}$$

where u_0 and ϱ are given operators, $[\cdot,\cdot]$ denotes commutator between operators and Δ is the usual Laplacian on $\mathbb{R}^d$. Roughly speaking, (1) can be seen as a large (or infinite) system of coupled Schrödinger equations. It is then an interesting problem to understand how the dispersive and regularity properties of the solutions to the Schrödinger equation translate to the solutions to the Liouville equation. Indeed, setting $\varrho = 0$ for the sake of concreteness, the solution u reads

$$u(t) = \sum_{j\in\mathbb{N}} \lambda_j |e^{it\Delta}u_j\rangle\langle e^{it\Delta}u_j| \qquad \text{if} \qquad u_0 = \sum_{j\in\mathbb{N}} \lambda_j |u_j\rangle\langle u_j|,$$

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where the λ_j are here positive numbers, the u_j are orthonormal in $L^2(\mathbb{R}^d)$, and we used the Dirac notation for the rank one projector $\varphi \mapsto u_j(u_j, \varphi) \equiv |u_j\rangle\langle u_j|\varphi\rangle$. The mathematical properties of the operator $e^{it\Delta}$ are well known, and the question is to understand how they translate to the collection defined by u. This has been an extensive subject of research, and some of the most notable results are the following:

– *The Lieb–Thirring inequalities [8,9].* They are the density operator counterpart of Gagliardo–Nirenberg–Sobolev inequalities: for u_0 as before, if we define formally the local density n_{u_0} and the local kinetic energy $\mathcal{E}_{u_0}$ by

$$n_{u_0} := \sum_{j\in\mathbb{N}} \lambda_j |u_j|^2, \qquad \mathcal{E}_{u_0} := \sum_{j\in\mathbb{N}} \lambda_j |\nabla u_j|^2,$$

an example of such inequalities is

$$\|n_{u_0}\|_{L^q(\mathbb{R}^d)} \leq C \left(\sum_{j\in\mathbb{N}} \lambda_j^p \right)^{\theta/p} \|\mathcal{E}_{u_0}\|_{L^1(\mathbb{R}^d)}^{1-\theta}$$

where $p \in [1, \infty]$ and

$$\theta = \frac{2p}{(d+2)p - d}, \qquad q = \frac{(d+2)p - d}{dp - (d-2)}.$$

– *The Strichartz inequalities [3,4].* They are generalizations to orthonormal systems of the classical Strichartz estimates for the Schrödinger equation, and read, see [4], Theorem 8,

$$\left\| \sum_{j\in\mathbb{N}} \lambda_j |e^{it\Delta} u_j|^2 \right\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \leq \left(\sum_{j\in\mathbb{N}} \lambda_j^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}},$$

with $p, q, d \geq 1$,

$$\frac{2}{p} + \frac{d}{q} = d, \qquad 1 \leq q < 1 + \frac{2}{d-1}.$$

These estimates on orthonormal systems have some remarkable properties: as mentioned in [3,4], if $\lambda_j = 0$ for $j > N$, then they behave much better in terms of N than the estimates obtained by applying the triangle inequality and the scalar estimates for the Schrödinger group $e^{it\Delta}$. This can be seen as a consequence of the orthogonality of the u_j, while that latter property is not used with the triangle inequality. Another interesting feature is the following: standard ways to define rigorously the local density n_{u_0} are either to assume that u_0 is trace class (i.e. $\sum_{j\in\mathbb{N}} \lambda_j < \infty$), or to assume that u_0 enjoys some smoothness, e.g.

$$\operatorname{Tr}\left((\mathbb{I} - \Delta)^{\beta/2} (u_0)^2 (\mathbb{I} - \Delta)^{\beta/2} \right)^p < \infty \tag{2}$$

for appropriate $p \geq 1$ and $\beta \geq 0$, see e.g. [7] (Tr above denotes the operator trace). The Strichartz estimates offer much weaker conditions that justify the definition of the local density: the local density of the solution u to the Liouville equation for $\varrho = 0$ is defined in the space $L_t^p L_x^q$ as soon as u_0 lies in Schatten space $\mathcal{J}_{2q/(q+1)}$. This can be seen as a combined effect of the orthogonality of the u_j and the dispersive effects of the Schrödinger operator. The situation is similar when $\varrho \neq 0$.