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Continuation and bifurcations of concave central configurations in the four and five body-problems for homogeneous force laws

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The central configurations given by an equilateral triangle and a regular tetrahedron with equal masses at the vertices and a body at the barycenter have been widely studied in [9] and [14] due to the phenomena of bifurcation occurring when the central mass has a determined value m^*. We propose a variation of this problem setting the central mass as the critical value m^* and letting a mass at a vertex to be the parameter of bifurcation. In both cases, 2D and 3D, we verify the existence of bifurcation, that is, for a same set of masses we determine two new central configurations. The computation of the bifurcations, as well as their pictures have been performed considering homogeneous force laws with exponent $a < -1$.

1. Introduction

In the classical Newtonian n-body problem the unique explicit solutions which are known until now are the homographic solutions, characterized by the fact that their configurations are invariant up to rotations and scaling, and in which each body describes a Keplerian orbit. These particular solutions are generated by initial configurations called *central configurations* (see [15] for details) and are, certainly the most celebrated of them. More precisely, let be $\mathbb{E}$ a finite dimensional Euclidean vector space; $\mathbf{q}_1, \cdots, \mathbf{q}_n \in \mathbb{E}$ the position vectors; $m_1, \cdots, m_n$ the masses given by n positives numbers; a a negative real number; $M = \sum m_i$ the

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total mass of the system and $\mathbf{q}_G = \frac{1}{M}\sum m_i\mathbf{q}_i$ the center of mass of the system, we can then give the following definition:

Definition 1. A configuration $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_n) \in \mathbb{E}^n$ is a *central configuration* (cc by short) for the masses $m_1, \cdots, m_n$ if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\lambda(\mathbf{q}_i - \mathbf{q}_G) = \sum_{j\neq i} m_j \|\mathbf{q}_i - \mathbf{q}_j\|^{2a}(\mathbf{q}_i - \mathbf{q}_j), \qquad \forall\ i = 1, \cdots, n. \tag{1}$$

When $a = -3/2$ we have the Newtonian case and when $a = -1$ the vortex case.

Technically, central configurations are zeros of a system of n equations with n vectorial variables and n positive parameters. In some special cases, e.g., when all masses are equal, some quite simple solutions having well-defined positions can be obtained in a trivial way. In effect, we can verify easily that any regular n-gon is a central configuration. If we add any additional mass at the origin, we still get a central configuration. A regular simplex of n points on an affine subspace $n-1$ dimensional is a central configuration no matter the values of the masses at the vertices. Other symmetric configurations like rhombus, kites and pyramids exist for systems with some equal masses [3,4,6,7,11,13,15].

As zeros of systems of equations with many parameters, it is expected that bifurcation phenomena arise. This happens if the Jacobian of the system becomes degenerate for some values of the parameters at a given trivial solution. This type of question has been approached in many works. In [9], the authors proved the existence of bifurcations at the neighborhood of the following planar central configurations: the equilateral triangle with equal masses at the vertices and a fourth mass m^* at the barycenter and the square with equal masses at the vertices and a fifth mass m^{**} at the center. In [8] Meyer studied the continuation of central configurations from the restricted $(3+1)$-body problem with two equal masses $1-\mu$ and a third mass 2μ forming an equilateral triangle to the full 4-body problem. He proved that for small values of the fourth mass, there are central configurations degenerated which undergo bifurcation for specific values of the parameter μ. The same bifurcation analysis was applied in [14] to show the existence of four branches of central configurations which arise from the regular tetrahedron with a critical mass at the barycenter. In [12], by using the S_4-equivariance of equations defining Dziobek's configurations, three new branches of bifurcations were found improving the previous result.

In this paper we investigate bifurcations arising from two concave central configurations in the 4 and 5-body problem. In the former case, we consider the equilateral triangle with masses $m_1 = m_2 = m_4 = 1$ at the vertices and a mass $m_3 = m$ at the barycenter. For any non-negative value of m, this is a central configuration. In [10], Palmore showed that $m^* = \frac{81+64\sqrt{3}}{249}$ is the unique value of m for which, this central configuration is degenerate. Unlike the analysis performed by Schmidt and Meyer [9], which have considered the mass at the barycenter as the bifurcation parameter, we set $m_4 = 1+\varepsilon$ and $m = m^*$, so that at $\varepsilon = 0$ we verify that the system of equations undergo a bifurcation producing symmetric and non symmetric central configurations for ε near zero. The S_2-equivariance of the equations and the Implicit Function Theorem (IFT for shorting) are applied in a singular case as in [5], and they will be the main tools in the proof of existence. For $m \neq m^*$ the equilateral triangle is non-degenerate and so, it can be continued in a neighborhood of $\varepsilon = 0$ as a family of isosceles triangles.

In the five body problem, we consider the regular tetrahedron with masses $m_1 = m_2 = m_3 = m_5 = 1$ at the vertices and a mass $m_4 = m$ at the barycenter. By setting $m_4 = m^{**} = \frac{10368+1701\sqrt{6}}{54952}$ as in [14], we proceed to the bifurcation analysis in a similar way, as in the former case, but taking into account that the equations present an S_3-equivariance. The fact that in this case one has a biggest system entails some extra difficulties, however still in this case, the equations present S_3 instead S_2-equivariance, so the computations can be reduced significantly and allows us to show the existence of several branches of central configurations emanating from the tetrahedron.