



On some canonical classes of cubic–quintic nonlinear Schrödinger equations



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ARTICLE INFO

Article history:

Received 12 October 2015

Available online 23 September 2016

Submitted by S.G. Krantz

Keywords:

Nonlinear Schrödinger

Lie symmetry

Blow-up

ABSTRACT

In this paper we bring into attention variable coefficient cubic–quintic nonlinear Schrödinger equations which admit Lie symmetry algebras of dimension four. Within this family, we obtain the reductions of canonical equations of nonequivalent classes to ordinary differential equations using tools of Lie theory. Painlevé integrability of these reduced equations is investigated. Exact solutions through truncated Painlevé expansions are achieved in some cases. One of these solutions, a conformal-group invariant one, exhibits blow-up behavior in finite time in L_p , L_∞ norm and in distributional sense.

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1. Introduction

In this paper we aim at studying a class of cubic–quintic nonlinear Schrödinger (CQNLS) equations, given in the form

$$iu_t + u_{xx} + g(x, t)|u|^2u + q(x, t)|u|^4u + h(x, t)u = 0 \quad (1.1)$$

in which the complex coefficients g , q and h will be assumed to have some specific forms so that the equation under consideration admits a four-dimensional Lie symmetry algebra.

The motivation for this study comes from the recent work [17] on a general class of cubic–quintic nonlinear Schrödinger equations given as

$$iu_t + f(x, t)u_{xx} + k(x, t)u_x + g(x, t)|u|^2u + q(x, t)|u|^4u + h(x, t)u = 0. \quad (1.2)$$

We performed classification of this family of equations according to Lie symmetry algebras they can admit. There u is a complex-valued function, f is real-valued, k, g, q, h are complex-valued functions of the form

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Table 1
Four dimensional symmetry algebras and the coefficients in (1.1).

No	Algebra	g	q	h
L_1	$\{T, D_1, C_1, W\}$	$(g_1 + ig_2)\frac{1}{x}$	$(q_1 + iq_2)$	$(h_1 + ih_2)\frac{1}{x^2}$
L_2	$\{T, P, B, W\}$	$(g_1 + ig_2)$	$(q_1 + iq_2)$	ih_2
L_3	$\{P, B, D_2, W\}$	$(g_1 + ig_2)$	$(q_1 + iq_2)t$	$i\frac{h_2}{t}$
L_4	$\{P, B, C_2, W\}$	$(g_1 + ig_2)$	$(q_1 + iq_2)(1 + t^2)$	$i\frac{t+h_2}{2(1+t^2)}$

$k = k_1(x, t) + ik_2(x, t)$, $g(x, t) = g_1(x, t) + ig_2(x, t)$, $q(x, t) = q_1(x, t) + iq_2(x, t)$ and $h(x, t) = h_1(x, t) + ih_2(x, t)$ with the assumption that $g \neq 0$ or $q \neq 0$, that is, at least one of g_1, g_2, q_1, q_2 is different from zero. Eq. (1.2) contains two physically important equations: cubic Schrödinger equation for $k = q = 0$ and quintic Schrödinger equation for $k = g = 0$ in one space dimension.

In [17] we transformed (1.2) to (1.1) by point transformations. Therefore (1.1) appears as a canonical equation when classifying the family (1.2) with respect to Lie symmetries they can admit. We would like to mention main results of this paper. We showed that the symmetry algebra L of Eq. (1.1) (equivalently, of Eq. (1.2)) is at most six-dimensional, that is, $1 \leq \dim L \leq 6$. The following results concern the canonical equation (1.1), therefore they actually stand for the (in look) more general family (1.2).

- R1. Any CQNLS equation within class (1.1) having a 6-dimensional symmetry algebra is equivalent to the quintic constant-coefficient equation with $q = q_1 + iq_2$, $g = h = 0$.
- R2. Any CQNLS equation within class (1.1) having a 5-dimensional symmetry algebra is equivalent to the cubic constant-coefficient equation with $g = g_1 + ig_2$, $q = h = 0$.
- R3. The symmetry algebra of the genuine (g and q not both zero) variable coefficient CQNLS equation can be at most 4-dimensional. There are precisely four inequivalent classes of equations in this case.
- R4. None of these classes can be transformed to the standard constant-coefficient cubic-quintic equation with $g = g_1 + ig_2$, $q = q_1 + iq_2$, $h = 0$.

According to these results, when a variable coefficient CQNLS equation has a 5- or 6-dimensional Lie symmetry algebra, it can be transformed to well-known constant coefficient cubic or quintic NLS equations, respectively. In the case when a variable coefficient CQNLS equation has a 4-dimensional symmetry algebra, we encountered four different classes of equations summarized in Table 1.

The basis elements of subalgebras are given as follows:

$$T = \partial_t, \quad P = \partial_x, \quad W = \partial_\omega, \quad B = t\partial_x + \frac{1}{2}x\partial_\omega, \quad (1.3)$$

$$D_1 = x\partial_x + 2t\partial_t - \frac{1}{2}\rho\partial_\rho, \quad D_2 = \frac{1}{2}x\partial_x + t\partial_t - \frac{1}{2}\rho\partial_\rho, \quad (1.4)$$

$$C_1 = xt\partial_x + t^2\partial_t - \frac{1}{2}t\rho\partial_\rho + \frac{1}{4}x^2\partial_\omega, \quad C_2 = xt\partial_x + (1 + t^2)\partial_t - t\rho\partial_\rho + \frac{1}{4}x^2\partial_\omega. \quad (1.5)$$

Here $u : \mathbb{R}^2 \rightarrow \mathbb{C}$ is expressed in terms of the modulus and the phase

$$u(x, t) = \rho(x, t)e^{i\omega(x, t)}. \quad (1.6)$$

Equations (1.1) with coefficients from Table 1 are representatives of class of equations which admit non-isomorphic four-dimensional symmetry algebras. If we are to say that the symmetry algebras do not extend to five- or six-dimensional ones at all, we have to put the conditions $(g, h) \neq (0, 0)$ for L_1 , $(g, h_2) \neq (0, 0)$ or $(q, h_2) \neq (0, 0)$ for L_2 , $(g, h_2) \neq (0, \frac{1}{4})$ or $(q, h_2) \neq (0, \{0, \frac{1}{2}\})$ for L_3 and $(g, h_2) \neq (0, 0)$ for L_4 [17]. Let us note that L_1 is non-solvable, L_2 is nilpotent, L_3 and L_4 are solvable algebras. Besides, L_1 is decomposable whereas the others are not. Eq. (1.1) with coefficients from Table 1 cannot be transformed to a constant-coefficient equation, therefore they will be the main subject of this article.

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