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Existence and uniqueness of solutions for a class of doubly degenerate parabolic equations

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ABSTRACT

In this paper, we study the Dirichlet problem for degenerate parabolic equations of the form $\frac{\partial u}{\partial t} - \mathrm{div}(a(x,t,u,\nabla u)) - \mathrm{div}\phi(u) = f - \mathrm{div}g$, where the main operator $a(x,t,u,\nabla u)$ is allowed to degenerate with respect to the unknown u. Under suitable conditions on a and ϕ, we prove that there exists an unique solution u of this problem.

1. Introduction

In this paper, we shall study the following degenerate parabolic problems:

$$(\mathscr{P})\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}(a(x,t,u,\nabla u)) - \mathrm{div}\phi(u) = f - \mathrm{div}g & \text{in } Q_T = \Omega\times(0,T), \\ u = 0 & \text{on } \Gamma = \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$, $T > 0$ is a given number, f, g and u_0 are measurable functions. Moreover, $a(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, and the function ϕ is continuous on $\mathbb{R}$ with values in $\mathbb{R}^N$. We assume that there exists a continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = 0$, such that $a(x,t,s,\xi) \geq \alpha(|s|)|\xi|^p$ for any $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and a.e. $(x,t) \in Q_T$. So problem $(\mathscr{P})$ is degenerate for the subset $\{(x,t) \in Q_T : u(x,t) = 0 \text{ or } \nabla u = 0\}$ of Q_T. We remark also that the principal part $\mathrm{div}(a(x,t,u,\nabla u))$ may be degenerate where u tends to infinite.

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The simplest model is the following problem involving the classical porous medium-type equation:

$$(P_0)\begin{cases} \frac{\partial u}{\partial t} - \triangle(|u|^{m-1}u) = f & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma, \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

which has been widely studied in the literature (see [20] and references therein). We remark that the problem we study here, has important and extensive applications in fluid dynamics in porous media, petroleum engineering, hydrology, etc. (see [11,12,20]).

In case of $\alpha \equiv$ constant > 0, existence and uniqueness results of problem $(\mathscr{P})$ have been well studied in the literature. Without the aim to be complete, we only mention the works of [1,2,22], where the existence of bounded weak solutions and renormalized solutions is proved. Especially, considering the case $f \in L^1(Q_T)$ and $g = 0$, the authors in [1] have studied the existence and uniqueness results for a class of Stefan problems (i.e. the time derivative term is $\frac{\partial b(u)}{\partial t}$, where b is a maximal monotone graph on $\mathbb{R}$).

In the case that α is a positive function, $p = 2$ and $g = 0$, an existence result for problem $(\mathscr{P})$ which involves a first order term, was proved in [4] (see also [8]), under some suitable assumptions on the data u_0 and f.

Considering the case that $\alpha(0) = 0$, $g = 0$ and $(\mathscr{P})$ involves a lower order term, Rakotoson studied the stationary case of $(\mathscr{P})$ in [19], and proved the existence of a weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In [13], Giachetti and Maroscia proved the existence of distributional solutions, with assumptions that f, u_0 are nonnegative functions satisfying $f \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega)$. As references for other estimates of solutions for the doubly degenerate equations, we only mention the work of [6,10,21] without the aim to be complete.

In the case that $g = 0$, $\alpha(0) = 0$ and $f \in L^\infty(Q_T)$, the existence of distributional nonnegative solutions to problem $(\mathscr{P})$ was investigated in [13]. Suppose that $u_0 \in L^\infty(\Omega)$ and $f \in L^q(Q_T)$ with $q > \max\{1 + \frac{N}{p}, 2\}$, the existence of bounded solutions to problem $(\mathscr{P})$ has been studied in [23]. As problem $(\mathscr{P})$ involves a lower order term, it is proved in [25] that there exists at least one renormalized solution for L^1 data. For the case that the time derivative term is $\frac{\partial b(u)}{\partial t}$, the authors proved that there exist bounded solutions in [26].

However, to the best of our knowledge, there are no existence and uniqueness results concerning degenerate problem $(\mathscr{P})$ for $g \neq 0$. Recently, the existence of bounded solutions to the stationary case of $(\mathscr{P})$ was proved in [24], under suitable integrability conditions on f and g. Motivated by the ideas in [1,8,13,19,24], here we shall deal with the case of $g \neq 0$ and remove the restriction that $\frac{1}{\alpha} \in L^1_{loc}[0, +\infty)$ in [19,25]. Furthermore, we also show that the solution u belongs to $C([0,T]; L^2(\Omega))$, and establish the uniqueness results under some additional assumptions (see Theorem 2.2).

We remark that when dealing with the existence result, one cannot expect that the solution u of $(\mathscr{P})$ belongs to $L^p(0,T; W_0^{1,p}(\Omega))$ for $g \neq 0$, due to the fact that the main operator is degenerate for the subset $\{(x,t) \in Q_T : u(x,t) = 0\}$. But instead, we only have $\tilde{\alpha}(u) \in L^p(0,T; W_0^{1,p}(\Omega))$, where $\tilde{\alpha}$ is defined in (2.3). To overcome this difficulty, we will prove the existence results to $(\mathscr{P})$ by using approximations twice, as in [24]. More precisely, we shall first prove the existence results for the case div$g \in L^1(Q_T)$. To do this, we shall find a priori estimates on the solutions u_n, where u_n are the solution of the approximating problems whose principal parts are uniformly coercive operators. Then one has to show that, up to subsequences, u_n converges to some function u a.e. in Q_T and $u_n|_{t=0}$ converges to $u|_{t=0}$. Due to the fact that problem $(\mathscr{P})$ is degenerate, we can not obtain the uniform estimates of u_n in $L^p(0,T; W_0^{1,p}(\Omega))$. Therefore the usual compactness result of Aubin type lemma can not be applied directly here. To this aim, we shall employ a test function method, and prove that u_n converges strongly to u in $C([0,T]; L^2(\Omega))$ (see step 4 in the proof of Theorem 3.1), which is the most technical part of the existence theorem. Moreover, we can not expect that ∇u_n converges to ∇u a.e. in Q_T and strongly in $(L^p(Q_T))^N$, but instead, we show that $\chi_{\{|u_n|>k\}} \nabla u_n$ converges to $\chi_{\{|u|>k\}} \nabla u$ strongly in $(L^p(Q_T))^N$ for any fixed $k > 0$.