J. Math. Anal. Appl. 446 (2017) 1–17

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On Valdivia strong version of Nikodym boundedness property [☆]

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ARTICLE INFO

Article history:
Received 16 October 2015
Available online 22 August 2016
Submitted by B. Cascales

Dedicated to the memory of Professor Manuel Valdivia (1928–2014)

Keywords:
Finitely additive scalar measure
(LF)-space
Nikodym and strong Nikodym property
Increasing tree
σ-algebra
Vector measure

ABSTRACT

Following Schachermayer, a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of Ω is said to have the *N-property* if a $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is uniformly bounded on $\mathcal{A}$, where $ba(\mathcal{A})$ is the Banach space of the real (or complex) finitely additive measures of bounded variation defined on $\mathcal{A}$. Moreover $\mathcal{B}$ is said to have the *strong N-property* if for each increasing countable covering $(\mathcal{B}_m)_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has the N-property. The classical Nikodym–Grothendieck's theorem says that each σ-algebra $\mathcal{S}$ of subsets of Ω has the *N-property*. The Valdivia's theorem stating that each σ-algebra $\mathcal{S}$ has the strong N-property motivated the main measure-theoretic result of this paper: We show that if $(\mathcal{B}_{m_1})_{m_1}$ is an increasing countable covering of a σ-algebra $\mathcal{S}$ and if $(\mathcal{B}_{m_1,m_2,\ldots,m_p,m_{p+1}})_{m_{p+1}}$ is an increasing countable covering of $\mathcal{B}_{m_1,m_2,\ldots,m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p$, then there exists a sequence $(n_i)_i$ such that each $\mathcal{B}_{n_1,n_2,\ldots,n_r}$, $r \in \mathbb{N}$, has the strong N-property. In particular, for each increasing countable covering $(\mathcal{B}_m)_m$ of a σ-algebra $\mathcal{S}$ there exists $\mathcal{B}_n$ which has the strong N-property, improving mentioned Valdivia's theorem. Some applications to localization of bounded additive vector measures are provided.

1. Introduction

Let $\mathcal{B}$ be a subset of an algebra $\mathcal{A}$ of subsets of a set Ω (in brief, set-algebra $\mathcal{A}$). The normed space $L(\mathcal{B})$ is the $span\{\chi_C : C \in \mathcal{B}\}$ of the characteristic functions of each set $C \in \mathcal{B}$ with the supremum norm $\|\cdot\|$ and $ba(\mathcal{A})$ is the Banach space of finitely additive measures on $\mathcal{A}$ with bounded variation endowed with the variation norm, i.e., $|\cdot| := |\cdot|(\Omega)$. If $\{C_i : 1 \leqslant i \leqslant n\}$ is a measurable partition of $C \in \mathcal{A}$ and $\mu \in ba(\mathcal{A})$ then $|\mu|(C) = \Sigma_i |\mu|(C_i)$ and, as usual, we represent also by μ the linear form in $L(\mathcal{A})$ determined by

☆ This research was supported for the first named author by the GAČR project 16-34860L and RVO: 67985840. It was also supported for the first and second named authors by Generalitat Valenciana, Conselleria d'Educació i Esport, Spain, Grant PROMETEO/2013/058.
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http://dx.doi.org/10.1016/j.jmaa.2016.08.032
0022-247X/

$\mu(\chi_C) := \mu(C)$, for each $C \in \mathcal{A}$. By this identification we get that the dual of $L(\mathcal{A})$ with the dual norm is isometric to $ba(\mathcal{A})$ (see e.g., [2, Theorem 1.13]).

Polar sets are considered in the dual pair $< L(\mathcal{A}), ba(\mathcal{A}) >$, M° means the polar of a set M and if $\mathcal{B} \subset \mathcal{A}$ the topology in $ba(\mathcal{A})$ of pointwise convergence in $\mathcal{B}$ is denoted by $\tau_s(\mathcal{B})$. $(E', \tau_s(E))$ is the vector space of all continuous linear forms defined on a locally convex space E endowed with the topology $\tau_s(E)$ of the pointwise convergence in E. In particular, the topology $\tau_s(L(\mathcal{A}))$ in $ba(\mathcal{A})$ is $\tau_s(\mathcal{A})$.

The convex (absolutely convex) hull of a subset M of a topological vector space is denoted by $co(M)$ ($absco(M)$) and $absco(M) = co(\cup\{rM : |r| = 1\})$. An equivalent norm to the supremum norm in $L(\mathcal{A})$ is the Minkowski functional of $absco(\{\chi_C : C \in \mathcal{A}\})$ ([14, Propositions 1 and 2]) and its dual norm is the $\mathcal{A}$-supremum norm, i.e., $\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$. The closure of a set is marked by an overline, hence if $P \subset L(\mathcal{A})$ then $\overline{span(P)}$ is the closure in $L(\mathcal{A})$ of the linear hull of P. $\mathbb{N}$ is the set $\{1, 2, \dots\}$ of positive integers.

Recall the classical Nikodym–Dieudonné–Grothendieck theorem (see [1, page 80, named as Nikodym–Grothendieck boundedness theorem]): *If $\mathcal{S}$ is a σ-algebra of subsets of a set Ω and M is a $\mathcal{S}$-pointwise bounded subset of $ba(\mathcal{S})$ then M is a bounded subset of $ba(\mathcal{S})$ (i.e., $\sup\{|\mu(C)| : \mu \in M, C \in \mathcal{S}\} < \infty$,* or, equivalently, $\sup\{|\mu|(\Omega) : \mu \in M\} < \infty$). This theorem was firstly obtained by Nikodym in [11] for a subset M of countably additive complex measures defined on $\mathcal{S}$ and later on by Dieudonné for a subset M of $ba(2^\Omega)$, where 2^Ω is the σ-algebra of all subsets of Ω, see [3].

It is said that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω has the *Nikodym property*, N-property in brief, if the Nikodym–Dieudonné–Grothendieck theorem holds for $\mathcal{B}$, i.e., *if each $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$* (see [12, Definition 2.4] or [15, Definition 1]). Let us note that in this definition we may suppose that M is $\tau_s(\mathcal{A})$-closed and absolutely convex. If $\mathcal{B}$ has N-property then the polar set $\{\chi_C : C \in \mathcal{B}\}^\circ$ is bounded in $ba(\mathcal{A})$, hence $\{\chi_C : C \in \mathcal{B}\}^{\circ\circ} = \overline{absco\{\chi_C : C \in \mathcal{B}\}}$ is a neighborhood of zero in $L(\mathcal{A})$, whence $L(\mathcal{B})$ is dense in $L(\mathcal{A})$.

It is well known that *the algebra of finite and co-finite subsets of $\mathbb{N}$ fails N-property* [2, Example 5 in page 18] and that Schachermayer proved that *the algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has N-property* (see [12, Corollary 3.5] and a generalization in [4, Corollary]). A recent improvement of this result for the algebra $\mathcal{J}(K)$ of Jordan measurable subsets of a compact k-dimensional interval $K := \Pi\{[a_i, b_i] : 1 \leqslant i \leqslant k\}$ in $\mathbb{R}^k$ has been provided in [15, Theorem 2], where Valdivia proved that *if $\mathcal{J}(K)$ is the increasing countable union $\cup_m \mathcal{B}_m$ there exists a positive integer n such that $\mathcal{B}_n$ has N-property* (see [8, Theorem 1] for a strong result in $\mathcal{J}(K)$). This fact motivated to say that a subset $\mathcal{B}$ of a set-algebra $\mathcal{A}$ has the *strong Nikodym property*, sN-property in brief, if for each increasing covering $\cup_m \mathcal{B}_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has N-property. As far as we know this result suggested the following very interesting Valdivia's open question (2013):

Problem 1 *([15, Problem 1]).* Let $\mathcal{A}$ be an algebra of subsets of Ω. Is it true that N-property of $\mathcal{A}$ implies sN-property?

Note that the Nikodym–Dieudonné–Grothendieck stating that every σ-algebra $\mathcal{S}$ of subsets of a set Ω has property N is a particular case of the following Valdivia's theorem.

Theorem 1 *([14, Theorem 2]). Each σ-algebra $\mathcal{S}$ of subsets of Ω has sN-property.*

Following [7, Chapter 7, 35.1] a family $\{B_{m_1,m_2,\dots,m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ of subsets of A is an *increasing web in A* if $(B_{m_1})_{m_1}$ is an increasing covering of A and $(B_{m_1,m_2,\dots,m_p,m_{p+1}})_{m_{p+1}}$ is an increasing covering of $B_{m_1,m_2,\dots,m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p$. We will say that *a set-algebra $\mathcal{A}$ of subsets of Ω has the web strong N-property* (*web-sN-property*, in brief) *if for each increasing web $\{\mathcal{B}_{m_1,m_2,\dots,m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ in $\mathcal{A}$ there exists a sequence $(n_i)_i$ in $\mathbb{N}$ such that each $\mathcal{B}_{n_1,n_2,\dots,n_i}$ has sN-property, for each $i \in \mathbb{N}$.*