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Decompositions of preduals of JBW and JBW* algebras ☆

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ABSTRACT

We prove that the predual of any JBW*-algebra is a complex 1-Plichko space and the predual of any JBW-algebra is a real 1-Plichko space. I.e., any such space has a countably 1-norming Markushevich basis, or, equivalently, a commutative 1-projectional skeleton. This extends recent results of the authors who proved the same for preduals of von Neumann algebras and their self-adjoint parts. However, the more general setting of Jordan algebras turned to be much more complicated. We use in the proof a set-theoretical method of elementary submodels. As a byproduct we obtain a result on amalgamation of projectional skeletons.

1. Introduction and main results

The aim of the present paper is to show that the predual of any JBW-algebra is 1-Plichko (i.e., it has a countably 1-norming Markushevich basis or, equivalently, it admits a commutative 1-projectional skeleton) and the same holds also for preduals of JBW^*-algebras. This extends previous results of the authors who showed in [4] the same statements on preduals of von Neumann algebras and their self-adjoint parts. JBW^*-algebras can be viewed as a generalization of von Neumann algebras, this class was introduced and studied in [10]; a JBW-algebra can be represented as the self-adjoint part of a JBW^*-algebra (see [10]). Precise definitions and a necessary background on these algebras are given in Section 2 below.

1-Plichko spaces form one of the largest classes of Banach spaces which admit a reasonable decomposition to separable pieces. This class and some related classes of Banach spaces together with the associated classes of compact spaces were thoroughly studied for example in [22,23,15]. The class of 1-Plichko spaces can be viewed as a common roof of previously studied classes of weakly compactly generated spaces [2], weakly

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K-analytic Banach spaces [21], weakly countably determined (Vašák) spaces [24,20] and weakly Lindelöf determined spaces [3]. Examples of 1-Plichko spaces include L^1 spaces, order continuous Banach lattices, spaces $C(G)$ for a compact abelian group G [16]; preduals of von Neumann algebras and their self-adjoint parts [4].

Let us continue by defining 1-Plichko spaces and some related classes. We will do it using the notion of a projectional skeleton introduced in [18]. If X is a Banach space, a *projectional skeleton* on X is an indexed system of bounded linear projections $(P_\lambda)_{\lambda\in\Lambda}$ where Λ is an up-directed set such that the following conditions are satisfied:

(i) $\sup_{\lambda\in\Lambda}\|P_\lambda\|<\infty$,
(ii) $P_\lambda X$ is separable for each λ,
(iii) $P_\lambda P_\mu = P_\mu P_\lambda = P_\lambda$ whenever $\lambda\le\mu$,
(iv) if (λ_n) is an increasing sequence in Λ, it has a supremum $\lambda\in\Lambda$ and $P_\lambda[X]=\overline{\bigcup_n P_{\lambda_n}[X]}$,
(v) $X=\bigcup_{\lambda\in\Lambda}P_\lambda[X]$.

The subspace $D=\bigcup_{\lambda\in\Lambda}P_\lambda^*[X^*]$ is called the *subspace induced by the skeleton*. If $\|P_\lambda\|=1$ for each $\lambda\in\Lambda$, the family $(P_\lambda)_{\lambda\in\Lambda}$ is said to be 1-*projectional skeleton*. The skeleton $(P_\lambda)_{\lambda\in\Lambda}$ is said to be *commutative* if $P_\lambda P_\mu = P_\mu P_\lambda$ for any $\lambda,\mu\in\Lambda$. A Banach space having a commutative (1-)projectional skeleton is called *(1-)Plichko*.

This is not the original definition used in [15,16] which says that X is (1-)Plichko if X^* admits a (1-)norming Σ-subspace. Let us recall that a subspace $D\subset X^*$ is r-norming ($r\ge 0$) if the formula

$$|x| = \sup\{|x^*(x)| : x^*\in D, \|x^*\|\le 1\}$$

defines an equivalent norm on X for which $\|\cdot\|\le r|\cdot|$.

Further, a subspace $D\subset X^*$ is a Σ-*subspace* of X^* if there is a linearly dense set $M\subset X$ such that

$$D=\{x^*\in X^* : \{m\in M : x^*(m)\neq 0\} \text{ is countable}\}.$$

It follows from [18, Proposition 21 and Theorem 27] that a norming subspace of X^* is a Σ-subspace of X^* if and only if it is induced by a commutative projectional skeleton, therefore our definitions are equivalent to the original ones.

Finally, recall that a Banach space X is called *weakly Lindelöf determined* (shortly *WLD*) if X^* is a Σ-subspace of itself or, equivalently, if X^* is induced by a commutative projectional skeleton in X.

Now we can formulate our main results. The following theorem extends [4, Theorems 1.1 and 1.4] to the more general setting of Jordan algebras. Precise definitions of the respective algebras are in the following section.

Theorem 1.1.

- *Let $\mathscr{M}$ be any JBW^*-algebra. Its predual $\mathscr{M}_*$ is a (complex) 1-Plichko space. Moreover, $\mathscr{M}_*$ is WLD if and only if $\mathscr{M}$ is σ-finite. In this case it is even weakly compactly generated.*
- *Let $\mathscr{M}$ be any JBW-algebra. Its predual $\mathscr{M}_*$ is a (real) 1-Plichko space. Moreover, $\mathscr{M}_*$ is WLD if and only if $\mathscr{M}$ is σ-finite. In this case it is even weakly compactly generated.*

As a corollary we get the following extension of a result of U. Haagerup [13, Theorem IX.1] on preduals of von Neumann algebras. It follows immediately from Theorem 1.1 and the definition of projectional skeletons. A Banach space X is said to have *separable complementation property* if each countable subset of X is contained in some separable complemented subspace of X.