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Exhaustion of an interval by iterated Rényi parking

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We study a variant of the Rényi parking problem in which car length is repeatedly halved and determine the rate at which the remaining space decays.

1. Introduction

Rényi modelled a long interval randomly jammed with cars of unit length. More precisely, each arriving car is parked at a location chosen according to uniform probability distribution over the set of available parking locations until no parking locations remain. The resulting gap lengths constitute a random variable with range $(0, 1)$. The constant $C_R \approx 0.747589$ is the limit, as the interval length tends to infinity, of the proportion of the interval covered and was analytically determined [5]. This Rényi parking constant is scale invariant, in the sense that it holds for cars of any fixed length. Variations on Rényi's parking problem consider higher dimensions, a discretised setting, or cars of mixed lengths. Modelling of physical processes, such as random sequential adsorption, frequently refer to the idealised Rényi model (see for example [3,4,6] and references therein). Here, we consider another variation. For our purposes it is more convenient to assume we have initially jammed with cars of length 2 leaving us with gaps distributed according to a probability distribution X_1 with range $(0, 2)$. In our formulation, the next step is that these X_1-distributed intervals are jammed by cars of length 1 to give a new gap distribution X_2 with range $(0, 1)$. We then introduce and jam with cars of length $\frac{1}{2}$ to get X_3 with range $(0, \frac{1}{2})$, then cars of length $\frac{1}{4}$ and so on. It is clear that our initial intervals are exhausted by this process. *Our motivating question is that of determining the rate of exhaustion.* We will show that, if L_n denotes the expected length of interval which remains uncovered after n stages, then L_{n+1}/L_n tends to a finite limit $R_{\frac{1}{2}} \approx 0.61$. We will numerically calculate this limit with error bounds and also see that the limiting behaviour is largely independent of the initial probability distribution X_1.

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In particular, X_1 need not be the Rényi distribution outlined above. Proof of these facts requires finding a limiting distribution X to the sequence $\{X_n\}$ and for this we need to normalise at each stage. Integral equations governing the evolution of $\{X_n\}$, their corresponding densities $\{f_n\}$, and steady state analogues will be found. We approach the solution to these equations by considering an infinite dimensional eigenvalue problem. Numerical analysis with error bounds shows that there is a unique eigenvalue of maximum absolute value. Convergence of the iterates of the linear system to the associated eigenspace yields convergence of the sequence of probability densities.

The paper is laid out as follows. In Section 2, we consider the sequence of cumulative probability distributions that arise as a result of the physical process of jamming with cars of fixed length at each stage before halving the car length for the next stage. This yields an integral equation relating the distribution at Stage $n+1$ to that at Stage n. We deduce from this the evolution equation in the case of a probability density which naturally leads to an integral equation governing any probability density which is a fixed point of this process. From this, we observe functional properties of any such fixed point which is sufficiently regular.

In Section 3, we use series representations of the iterated sequence of probability densities in order to recast the integral formulation of density evolution as an infinite linear system. The sought-after asymptotic decay rate $R_{\frac{1}{2}}$ is seen to be one half of the spectral radius of the infinite matrix and spectral properties of this matrix, $\mathbf{A}$, are explored.

Following mention of floating point concerns in Section 4, careful numerical analysis of a similarity transformation of the matrix $\mathbf{A}$ is provided in Section 5, based on Gershgorin's Theorem with error and truncation estimates. Using the same similarity transformation, we deduce a convergence result for iteration of ℓ^1 vectors under $\mathbf{A}$, corresponding to convergence of our probability densities.

Finally, in Section 6, we consider convergence for more general initial distributions without density.

2. The model

Let N be the large number of gaps in a long interval of length L formed, say, by parking cars of length 2 until jamming occurs. The gap lengths form a random variable, X_1, with range $(0, 2)$. The expected number of these gaps which have length greater than 1 is $NP(X_1 > 1)$. Each of these has a car of length 1 placed in it, randomly, by uniform distribution over the set of available locations. This creates $2NP(X_1 > 1)$ new gaps, referred to as Stage 2 gaps, of length in $(0, 1)$ and removes $NP(X_1 > 1)$ Stage 1 gaps of length in $(1, 2)$. The length of one of these $2NP(X_1 > 1)$ Stage 2 gaps is a random variable with range $(0, 1)$ which we call Y. The distribution of all $N + NP(X_1 > 1)$ gaps, whose length ranges over $(0, 1)$, will be represented by the random variable $\tilde{X}_2$ which we normalise to $X_2 = 2\tilde{X}_2$ with range $(0, 2)$.

We calculate the probability distribution of $\tilde{X}_2$. Observe that each gap at this point may have existed at the first stage or have been created on introduction of cars at the second stage. We refer to these as Stage 1 gaps and Stage 2 gaps respectively. For $t \in (0, 1)$,

$$P(\tilde{X}_2 < t) = P(\text{gap from Stage 1})P(X_1 < t|X_1 < 1) + P(\text{gap from Stage 2})P(Y < t). \tag{2.1}$$

Clearly,

$$P(\text{gap from Stage 1}) = \frac{N - NP(X_1 > 1)}{N + NP(X_1 > 1)} = \frac{1 - C_1}{1 + C_1}$$

where $C_1 = P(X_1 > 1)$ and

$$P(\text{gap from Stage 2}) = \frac{2C_1}{1 + C_1}.$$