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An example of a non-commutative uniform Banach group

Michal Doucha [1]

Institute of Mathematics, Polish Academy of Sciences, 00-656 Warszawa, Poland

A R T I C L E I N F O

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A B S T R A C T

Benyamini and Lindenstrauss mention in their monograph (Benyamini and Lindenstrauss, 2000 [2]) that there is no known example of a non-commutative uniform Banach group. Prassidis and Weston also asked whether there is a non-commutative example. We answer this problem affirmatively. We construct a non-commutative uniform Banach group which has the free group of countably many generators as a dense subgroup. Moreover, we show that our example is a free one-generated uniform Banach group whose metric induced by the norm is bi-invariant.

0. Introduction

A uniform Banach group is a Banach space equipped with an additional group structure so that the group unit coincides with the Banach space zero and the group operations are uniformly continuous with respect to norm. Uniform Banach groups were introduced and studied by Enflo in [4,5] with connection to the infinite-dimensional version of the Hilbert's fifth problem. Typical example comes when we are given two Banach spaces X and Y and a uniform homeomorphism $\phi : X \to Y$ between them such that $\phi(0) = 0$. Then we can define a (commutative) group operation $\cdot$ on X as follows: for $x, y \in X$ we set $x \cdot y = \phi^{-1}(\phi(x) + \phi(y))$. Note that unless ϕ is linear there is no a priori connection between the two group operations $+$ (resp. $+_X$) and $\cdot$.

A comprehensive source of information about uniform Banach groups is Chapter 17 in [2]. As mentioned there, the following problem was left open. *Does there exist a non-commutative uniform Banach group?* This question was also asked by Prassidis and Weston in [10] and [9]. Here we give a positive answer to this question. The following is the main result.

E-mail address: m.doucha@post.cz.

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Theorem 0.1. *There exists an infinite dimensional separable Banach space* $(\mathbb{X}, +, 0, \|\cdot\|)$ *equipped with an additional group structure* $(\cdot, ^{-1}, 0)$ *such that*

- *its unit coincides with the Banach space zero,*
- *its group multiplication* $\cdot$ *is invariant with respect to the norm* $\|\cdot\|$*, i.e.* $\|x - y\| = \|a \cdot x \cdot b - a \cdot y \cdot b\|$*, for all* $x, y, a, b \in X$*,*
- F_∞*, the free group of countably many generators, is a dense subgroup of* $(\mathbb{X}, \cdot, ^{-1}, 0)$*.*

In particular, there exists a non-commutative uniform Banach group.

1. Preliminaries

We assume the reader to know basic facts about uniform spaces. We refer to Chapter 8 in [6] for more information.

Recall that for any topological group G there are two distinguished compatible uniformities: the left uniformity $\mathcal{U}_L$ generated by basic entourages of the form $\{(g, h) : g^{-1}h \in U\}$, where U is a basic neighborhood of the identity in G; and the right uniformity $\mathcal{U}_R$ which is generated by basic entourages of the form $\{(g, h) : hg^{-1} \in U\}$, where U is again a basic neighborhood of the identity in G.

We start with the following definition given by Enflo in [4].

Definition 1.1. Let G be a topological group. G is called *uniform* if there exists a compatible uniformity on G such that the group multiplication is uniformly continuous with respect to that uniformity.

Below, we collect some basic facts about uniform groups.

Fact 1.2.

(1) (See Propositions 1.1.2 and 1.1.3 in [4]) A topological group G is uniform if and only if it has a unique compatible uniformity, which happens if and only if the left and right uniformities coincide.
(2) (Folklore) That is in turn equivalent with the fact that there exists a neighborhood basis of the unit of G consisting of open sets closed under conjugation. Such groups are more often called SIN *(small invariant neighborhood) groups, or also* balanced *groups.*
(3) (Folklore) In case that G is metrizable, i.e. the neighborhood basis can be taken countable, G is uniform, equivalently SIN, if and only if it admits a compatible bi-invariant metric; i.e. metric d such that for any $x, y, a, b \in G$ *it holds that* $d(x, y) = d(axb, ayb)$ *(the same reasoning gives that if G is a non-metrizable SIN group then its topology is given by a family of bi-invariant pseudometrics).*

Examples:

(a) All abelian and compact topological groups are uniform groups. For the abelian groups, it follows from the fact that the left and right uniformities are one and the same by commutativity. For compact groups, it suffices to take some compatible uniformity and notice that the group operations are uniformly continuous with respect to it by compactness.
(b) The Heisenberg group $UT_3^3(\mathbb{R})$ consisting of the upper triangular 3×3-matrices is not uniform. Note that the Heisenberg group is very close to both being abelian and compact. It is a locally compact group of nilpotency class 2. Since $UT_3^3(\mathbb{R})$ is metrizable it is sufficient to show that it does not admit a compatible bi-invariant metric. Suppose for contradiction that d is a compatible bi-invariant metric on $UT_3^3(\mathbb{R})$. Given matrices