J. Math. Anal. Appl. 446 (2017) 114–129

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some properties of the difference between the Ramanujan constant and beta function ☆

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ARTICLE INFO

Article history:
Received 12 June 2016
Available online 29 August 2016
Submitted by B.C. Berndt

Keywords:
The Ramanujan constant
Beta function
Monotonicity and convexity
Functional inequalities
Power series
The Riemann zeta function

ABSTRACT

The authors present the power series expansions of the function $R(a) - B(a)$ at $a = 0$ and at $a = 1/2$, show the monotonicity and convexity properties of certain familiar combinations defined in terms of polynomials and the difference between the so-called Ramanujan constant $R(a)$ and the beta function $B(a) \equiv B(a, 1-a)$, and obtain asymptotically sharp lower and upper bounds for $R(a)$ in terms of $B(a)$ and polynomials. In addition, some properties of the Riemann zeta function $\zeta(n)$, $n \in \mathbb{N}$, and its related sums are derived.

1. Introduction

For real numbers $x, y > 0$, the gamma, beta and psi functions are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,\ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

respectively. (Cf. [1,3,12,13].) Let $\gamma = 0.5772156649\cdots$ be the Euler constant. The so-called Ramanujan constant $R(a)$ is defined by

$$R(a) \equiv -2\gamma - \psi(a) - \psi(1-a) \quad \text{for} \quad a \in (0,1), \tag{1.2}$$

which is the special case of the following function of two parameters a and b

$$R(a,b) \equiv -2\gamma - \psi(a) - \psi(b) \quad \text{for} \quad a, b \in (0,\infty) \tag{1.3}$$

☆ This research is supported by the NSF of PR China (Grant No. 11171307, No. 11401531).
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http://dx.doi.org/10.1016/j.jmaa.2016.08.043
0022-247X/

when $b = 1 - a$. By [1, 6.3.4], $R(1/2) = \log 16$, and by the symmetry, we can sometimes assume that $a \in (0, 1/2]$ in (1.2).

For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) ={}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!}x^n, \ x \in (-1,1),$$

where (a, n) denotes the shifted factorial function $(a, n) = a(a + 1) \cdots (a + n - 1)$ for $n \in \mathbb{N}$, and $(a, 0) = 1$ for $a \neq 0$. $F(a, b; c; x)$ is said to be zero-balanced if $c = a+b$. The asymptotic properties of $F(a, b; a+b; x)$ as $x \to 1$ are related to $B(a, b)$ and $R(a, b)$. (See [1, 15.3.10], [2, Theorem 1.3 & 1.4] and [6,7,11].) For example, $F(a, b; a + b; x)$ satisfies the following S. Ramanujan's asymptotic relation (cf. [2, (1.6)])

$$B(a,b)F(a,b;a+b;x) + \log(1-x) = R(a,b) + O((1-x)\log(1-x)), \ x \to 1, \tag{1.4}$$

by which

$$\lim_{x\to 1^-} \frac{F(a,b;a+b;x)}{\log[1/(1-x)]} = \frac{1}{B(a,b)}. \tag{1.5}$$

(See also [5, Theorem 2.1.3].) For $a \in (0, 1/2]$, $r \in [0, 1]$ and $r' = \sqrt{1 - r^2}$, let $\mathscr{K}_a(r)$ and $\mathscr{K}_a'(r)$ denote the generalized elliptic integrals of the first kind, which are defined by

$$\mathscr{K}_a(r) = \frac{\pi}{2}F(a,1-a;1;r^2) \quad \text{and} \quad \mathscr{K}_a'(r) = \mathscr{K}_a(r') \quad \text{for} \quad r \in (0,1), \ \mathscr{K}_a(0) = \pi/2, \ \mathscr{K}_a(1) = \infty.$$

Then, by (1.4),

$$B(a)\mathscr{K}_a(r) - \pi \log \frac{e^{R(a)/2}}{r'} = O((r')^2 \log r'), \ r \to 1, \tag{1.6}$$

where

$$B(a) \equiv B(a,1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}. \tag{1.7}$$

For $a \in (0, 1/2]$, define the homeomorphisms $\mu_a : (0, 1) \to (0, \infty)$ and $\varphi_K^a : [0, 1] \to [0, 1]$ by

$$\mu_a(r) \equiv \frac{\pi}{2\sin(\pi a)}\frac{\mathscr{K}_a'(r)}{2\mathscr{K}_a(r)} = \frac{B(a)\mathscr{K}_a'(r)}{2\mathscr{K}_a(r)}$$

and

$$\varphi_K^a(r) \equiv \mu_a^{-1}(\mu_a(r)/K), \ \varphi_K^a(0) = \varphi_K^a(1) - 1 = 0,$$

respectively. Then the following Ramanujan's generalized modular equation with signature $1/a$ and order (or degree) p

$$\frac{F(a,1-a;1;1-s^2)}{F(a,1-a;1;s^2)} = p\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)}, \ 0 < r < 1,$$

and its solution s can be written as

$$\mu_a(s) = p\mu_a(r) \quad \text{and} \quad s = \varphi_{1/p}^a(r),$$

respectively.