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Homogenization of degenerate coupled fluid flows and heat transport through porous media

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A R T I C L E I N F O

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A B S T R A C T

We establish a homogenization result for a fully nonlinear degenerate parabolic system with critical growth arising from the heat and moisture flow through a partially saturated porous media. Existence of a global weak solution of the mesoscale problem is proven by means of a semidiscretization in time, a priori estimates and passing to the limit from discrete approximations. After that, porous material exhibiting periodic spatial oscillations is considered and the two-scale convergence (as the oscillation period vanishes) to a corresponding homogenized problem is rigorously proven.

1. Introduction

This paper deals with the two-scale homogenization of a class of doubly nonlinear degenerate parabolic problems arising from coupled transport processes in porous media.

Model problem Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the 1-dimensional measure of $\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)$ equals zero. Let $T \in (0, \infty)$ be fixed throughout the paper. Let us abbreviate $\Omega_T = \Omega \times (0, T)$, $\Gamma_{DT} = \Gamma_D \times (0, T)$ and $\Gamma_{NT} = \Gamma_N \times (0, T)$ and let $\mathcal{Y} = (0, 1)^2$ be a periodicity cell. Vectors and vector functions are denoted by boldface letters. Let $\varepsilon > 0$ be a *small* scalar parameter. From the geometrical point of view, ε is the characteristic length representing the small scale variability of the porous media. We study the homogenization of the doubly nonlinear degenerate system indexed by the scale parameter ε, namely,

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$$\partial_t b\left(x/\varepsilon, u^\varepsilon\right) + \nabla \cdot \boldsymbol{q}^\varepsilon = 0 \qquad \text{in } \Omega_T, \tag{1.1}$$

$$\partial_t \left[b\left(x/\varepsilon, u^\varepsilon\right)\theta^\varepsilon + \rho(x/\varepsilon)\theta^\varepsilon\right] = \nabla\cdot\left[\chi_2(x/\varepsilon)\lambda(\theta^\varepsilon, u^\varepsilon)\nabla\theta^\varepsilon - \theta^\varepsilon\boldsymbol{q}^\varepsilon\right] \qquad \text{in } \Omega_T, \tag{1.2}$$

$$u^\varepsilon = 0 \qquad \text{in } \Gamma_{DT}, \tag{1.3}$$

$$\theta^\varepsilon = 0 \qquad \text{in } \Gamma_{DT}, \tag{1.4}$$

$$\boldsymbol{q}^\varepsilon \cdot \boldsymbol{n} = 0 \qquad \text{in } \Gamma_{NT}, \tag{1.5}$$

$$\nabla\theta^\varepsilon \cdot \boldsymbol{n} = 0 \qquad \text{in } \Gamma_{NT}, \tag{1.6}$$

$$u^\varepsilon(0) = u_0 \qquad \text{in } \Omega, \tag{1.7}$$

$$\theta^\varepsilon(0) = \theta_0 \qquad \text{in } \Omega. \tag{1.8}$$

Here

$$\boldsymbol{q}^\varepsilon = -\left[\chi_1(x/\varepsilon)a(\theta^\varepsilon)\nabla u^\varepsilon + \boldsymbol{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon)\right]. \tag{1.9}$$

From the physical point of view, equations (1.1) and (1.2), respectively, represent the mass balance of moisture and the heat equation for the porous system after the so-called Kirchhoff transformation. Functions $u^\varepsilon : \Omega_T \to \mathbb{R}$ and $\theta^\varepsilon : \Omega_T \to \mathbb{R}$ are the unknowns. $\boldsymbol{n}$ is the outer unit normal vector to $\partial\Omega$, $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\boldsymbol{g} : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, $\chi_1, \chi_2 : \mathbb{R}^2 \to \mathbb{R}$, $\rho : \mathbb{R}^2 \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ and $\theta_0 : \Omega \to \mathbb{R}$ are given functions. Roughly speaking, the coefficient functions χ_1, χ_2 and ρ are $\mathcal{Y}$-periodic functions on $\mathbb{R}^2$ and similarly, b and $\boldsymbol{g}$ are $\mathcal{Y}$-periodic with respect to the first variable (see the next section for precise assumptions). As ε gets smaller, the coefficient functions χ_1, χ_2, ρ, b and $\boldsymbol{g}$ in (1.1)–(1.2) oscillate more rapidly. We solve the classical homogenization problem, namely investigate the behavior of the solution $[u^\varepsilon, \theta^\varepsilon]$ in the limit (as $\varepsilon \to 0$).

Remark 1.1. The analysis presented here can be straightforwardly extended to a setting with nonhomogeneous boundary conditions (see [9] for details). Here we work with homogeneous boundary conditions just to simplify and shorten the presentation and avoid unnecessary technicalities in the existence result.

System (1.1)–(1.2) can be seen as the special case of more general problem, the so-called doubly nonlinear problem, $\partial_t \mathbf{B}(x, \boldsymbol{u}) - \nabla \cdot \mathbf{A}(x, \boldsymbol{u}, \nabla\boldsymbol{u}) = \mathbf{0}$, with nonlinearities in both, parabolic as well as elliptic parts. From theoretical point of view, as far as we know, no general existence, uniqueness and regularity theory is developed for such kinds of problems. Partial results assuming special structure of operators $\mathbf{B}$ and $\mathbf{A}$ can be found e.g. in [4,18]. However, these results are not applicable if $\mathbf{B}$ does not take the subgradient structure, which is the case of the coupled system (1.1)–(1.2) due to a non-symmetry in the parabolic term. Moreover, under rather general assumptions on b, the equation (1.1) degenerates in the parabolic part. Finally, it is worth noting that further difficulty lies in the convective term in the heat equation, which represents strong nonlinearity in the model. Hence, we deal with the nonlinear degenerate system under critical growth assumptions. In the present paper we prove the existence of the weak solution to (1.1)–(1.9) and rigorously derive the corresponding homogenized problem letting $\varepsilon \to 0$ and using the two-scale convergence theory (in the sense of [39], see also [3]). As far as we know, this is the first attempt to carry out such an analysis for (1.1)–(1.9).

A brief bibliographical survey Qualitative properties of particular problems like (1.1), (1.3), (1.5) and (1.7), such as existence, uniqueness and regularity, have been studied by several authors, see e.g. [4,18]. More recently, in [11], the authors studied homogenization of the decoupled nonlinear degenerate parabolic