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Naively Haar null sets in Polish groups

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ABSTRACT

Let $(G, \cdot)$ be a Polish group. We say that a set $X \subset G$ is *Haar null* if there exists a universally measurable set $U \supset X$ and a Borel probability measure μ such that for every $g, h \in G$ we have $\mu(gUh) = 0$. We call a set X *naively Haar null* if there exists a Borel probability measure μ such that for every $g, h \in G$ we have $\mu(gXh) = 0$. Generalizing a result of Elekes and Steprāns, which answers the first part of Problem FC from Fremlin's list, we prove that in every abelian Polish group there exists a naively Haar null set that is not Haar null.

1. Introduction

Let $(G, \cdot)$ be a Polish group. It is well known that there exists a left Haar measure on G (that is, a regular left invariant Borel measure that is finite for compact sets and positive for non-empty open sets) if and only if G is locally compact. It can be proved that the ideal of left Haar measure zero sets does not depend on the choice of the measure, moreover, it coincides with the ideal of the right Haar null sets (that can be defined analogously). This ideal plays an important role in the study of locally compact groups and there are a lot of interesting non-locally compact groups, so it is very natural to try to construct well-behaved generalizations of this notion in non-locally compact groups.

Christensen [2] suggested a generalization, which is widely used in diverse areas of mathematics. We will call a set *universally measurable* if it is measurable with respect to every Borel probability measure and we identify Borel measures with their completions.

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Definition 1.1. A set $X \subset G$ is called *Haar null* if there exists a universally measurable set $U \supset X$ and a Borel probability measure μ on G such that $\mu(gUh) = 0$ for every $g, h \in G$.

For a Haar null set we will call a corresponding measure a *witness measure* and the set U its *universally measurable hull*. Notice that some authors use a slightly more restrictive notion, namely they require the hull to be a Borel set. These two notions provably differ in non-locally compact abelian Polish groups (see [6], where the above notion is called generalized Haar null), although in practice this makes little difference as the studied sets are typically Borel.

Christensen proved that in a locally compact Polish group a set is Haar null if and only if it is of measure zero with respect to a (or equivalently, every) Haar measure. He also showed that the collection of Haar null sets form a σ-ideal in every Polish group.

Our paper is motivated by the first part of Problem FC on Fremlin's list [8]. The problem is whether we really need the universally measurable hulls in this definition. Let us consider the following notion.

Definition 1.2. A set $X \subset G$ is called *naively Haar null* if there exists a Borel probability measure μ on G such that $\mu(gXh) = 0$ for every $g, h \in G$.

Using this terminology Fremlin's problem asks whether every naively Haar null set is Haar null. This question was answered by Elekes and Steprāns [5]. Notice that this question makes sense in any uncountable Polish group, though the original question was formulated in $\mathbb{R}$.

It was observed by Dougherty [4] that under the Continuum Hypothesis (CH) the answer is negative in the groups of the form $G \times G$. In fact, it is easy to see that if we consider a well-ordering $<_W$ of G in order type ω_1 as a subset W of $G \times G$ then both W and $(G \times G) \setminus W$ are naively Haar null. In particular, since $G \times G$ is clearly not Haar null and Haar null sets form a σ-ideal, we obtain that either W or its complement is a naively Haar null, non-Haar null set.

Elekes and Steprāns [5] proved that in $\mathbb{R}^n$ the assumption of CH can be dropped. In this paper we extend their result to every abelian Polish group, proving the following statement.

Theorem 1.3. *Let G be an uncountable abelian Polish group. There exists a subset of G that is naively Haar null but not Haar null.*

We have to treat the case of locally compact and non-locally compact Polish topological groups separately. We start with the locally compact case, which is essentially a transfinite construction, while to solve the non-locally compact case we use ideas from [6].

In fact, we will prove slightly more in both cases. A natural modification of the definition of Haar nullness that was investigated by several authors ([11,12] etc.) is the following:

Definition 1.4. A set $X \subset G$ is called *left Haar null* if there exists a universally measurable set $U \supset X$ and a Borel probability measure μ on G such that $\mu(gU) = 0$ for every $g \in G$.

The naive version of this notion can be defined analogously. It is easy to see using convolution that in locally compact groups a set is left Haar null if and only if it is of measure zero with respect to a Haar measure.

In the locally compact case we show that every Polish group has a naively left Haar null set that is not Haar null.

In the non-locally compact case our results (including of course the part cited from [6]) can be generalized to every non-locally compact Polish group that admits a two-sided invariant metric.