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Plateau's rotating drops and rotational figures of equilibrium

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ABSTRACT

We give a detailed classification of all rotationally symmetric figures of equilibrium corresponding to rotating liquid masses subject to surface tension. When the rotation rate is zero, these shapes were studied by Delaunay who found six different qualitative types of complete connected interfaces (spheres, cylinders, unduloids, nodoids, catenoids, and planes). We find twenty-six qualitatively different interfaces providing a complete picture of symmetric equilibrium shapes, some of which have been studied by other authors. In particular, combining our work with that of Beer, Chandrasekhar, Gulliver, Smith, and Ross, we conclude that every compact equilibrium is in either a smooth connected one parameter family of spheroids or a smooth connected one parameter family of tori (possibly immersed in either case).

1. Introduction

It is well known that an immiscible liquid drop immersed completely in a homogeneous fluid and isolated from body forces (magnetic, gravitational, etc.) assumes, at rest, the shape of a sphere. This fact was observed by Joseph Plateau in the 1840s while conducting experiments with neutrally buoyant oil drops immersed in a solution of alcohol and water. Plateau observed that such an isolated drop may rigidly rotate and assume an axisymmetric shape [9]. Based on his observations Plateau also conjectured that toroidal equilibrium shapes exist and satisfy an appropriate mathematical equation.

In 1855, August Beer [1] derived a geometric equation modeling the shape of rotationally symmetric rotating drops: The mean curvature of the boundary of the drop is a quadratic function of distance to the axis of rotation. Beer went on to study certain simply connected solutions of his equation. In 1914, Rayleigh [10] obtained sufficient conditions to guarantee the existence of a toroidal solution. In his 1964 paper [3], Chandrasekhar considered the stability of Beer's simply connected rotationally symmetric solutions.

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All the work mentioned above involves rotationally symmetric figures, but the geometric condition can also be applied to non-symmetric interfaces. Henry Wente [13] proved that every equilibrium rotating drop has a symmetry plane orthogonal to its axis of rotation, and later Rafael López [8] showed, under certain restrictions, that rotating bubbles must be axially symmetric. However, Brown and Scriven [2] obtained multi-lobed drop shapes numerically as bifurcations from the rotationally symmetric family. Two and three lobed shapes were produced experimentally in the low gravity environment of Spacelab by a group of researches from JPL [12]. A mathematical proof of existence for specific non-symmetric solutions of the modeling equation bifurcating from the rotationally symmetric solutions is not known. Kapouleas [7] has constructed non-symmetric solutions which share the symmetry of the surfaces obtained by Brown and Scriven but are far from the rotationally symmetric solutions. We note also that Wilkin-Smith [14] obtained an interesting existence result for solutions close to the sphere. These solutions may be non-symmetric, but the proof is non-constructive and apparently does not provide this information. The paper [14] also contains an extensive list of references related to rotating drops.

Plateau's conjecture on the existence of toroidal solutions was unresolved until 1984, when Gulliver [4] verified that toroidal rotating drops do indeed exist. Gulliver found a one parameter family of embedded tori each with convex cross section. Smith and Ross [11], following Gulliver, characterized all embedded toroidal figures of equilibrium. Hynd and McCuan showed in [6] the existence of infinitely many immersed toroidal solutions with figure-eight cross section.

Heine [5] has recently numerically computed non-symmetric toroidal shapes with many lobes. Like the shapes of Brown and Scriven, there is no rigorous mathematical proof that these shapes correspond to solutions of the modeling equation.

In this work, we classify all rotational figures of equilibrium. These include the classical rotating drops studied by Beer, Rayleigh, Chandrasekhar (spheroids), rotating bubbles, and toroidal drops. There are also unbounded solutions and additional immersed solutions which have received little attention in the literature but arise in modeling the interfaces of rotating liquid masses in contact with a rotating container; see Fig. 5. Our classification yields twenty-six qualitatively different shapes. We show that every compact rotational figure of equilibrium is a spheroid or toroidal figure and prove that all of these surfaces belong to two well-defined smooth one parameter families. We believe this work provides a complete picture for the rotationally symmetric equilibrium figures.

1.1. Summary of shapes

The complete list of qualitative types of rotational figures of equilibrium includes the Delaunay surfaces (for zero angular velocity) and the twenty-six surfaces we now describe. Perhaps the easiest way to describe the latter shapes is by comparison to the Delaunay surfaces shown in Fig. 1 and as deformations of surfaces in the family itself. We point out, in particular, the qualitative properties of the nodoid, having an immersed periodic meridian that loops toward the axis with no inflections, and the unduloid, having an embedded periodic meridian with one inflection in each half-period.

Circular cylinders (with the same axis as the axis of rotation) are possible solutions with both zero and nonzero angular velocity. These shapes of liquids in rotation are indistinguishable from the shapes of liquid cylinders at rest. Every possible radius is represented among the cylinders, and the orientation may be taken in either direction (to model liquid rotating within the cylinder or a cylinder enclosed by rotating liquid).

There are also solutions that are qualitatively the same as Delaunay unduloids; see Fig. 2(a). To denote the fact that these surfaces are analytically distinct from unduloids, we refer to them as *unduloid type*. In the case of zero angular velocity, i.e., the surfaces of Delaunay, the unduloids are often viewed as smooth deformations of cylinders under which necks and bulges appear. The unduloid type surfaces, for nonzero angular velocity, may also be viewed as deformations of cylindrical solutions.