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# Simple conditions for parametrically excited oscillations of generalized Mathieu equations

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A B S T R A C T

The following equation is considered in this paper:

$$x'' + (-\alpha + \beta \cos(\gamma t))x = 0,$$

where $\alpha$, $\beta$ and $\gamma$ are real parameters and $\gamma > 0$. This equation is referred to as Mathieu's equation when $\gamma = 2$. The parameters determine whether all solutions of this equation are oscillatory or nonoscillatory. Our results provide parametric conditions for oscillation and nonoscillation; there is a feature in which it is very easy to check whether these conditions are satisfied or not. Parametric oscillation and nonoscillation regions are drawn to help understand the obtained results.



## 1. Introduction

We consider the second-order differential equation

$$x'' + (-\alpha + \beta \cos(\gamma t))x = 0, \tag{1.1}$$

where the prime denotes $d/dt$; the parameters $\alpha$, $\beta$ and $\gamma$ are real numbers; $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma > 0$. A phenomenon where the amplitude is magnified by varying some parameters is called a *parametric excitation*. Equation (1.1) is a mathematical approximation model to describe the parametric excitation. As a familiar example of parametric excitation, we mention children playing on a swing. When children play on a swing, they move the center of gravity by periodically standing and squatting on the seat of the swing. Movement of the center of gravity amplifies the width of the swing's oscillation. This movement can be considered to cause the parameter variation.

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As a pioneer work of parametric excitation, Mathieu [18] has studied the vibration of the oval type drum film and derived the special case of equation (1.1) with $\gamma = 2$; namely,

$$x'' + (-\alpha + \beta\cos(2t))x = 0. \tag{1.2}$$

This equation was named Mathieu's equation after him. Mathieu's equation has been applied to many problems in physics and the natural sciences. For example, by the transformation from rectangular coordinates $(x, y)$ to elliptic coordinates $(\xi, \eta)$:

$$x = c\cosh\xi\cos\eta \quad \text{and} \quad y = c\sinh\xi\sin\eta,$$

the two-dimensional Helmholtz equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0$$

becomes

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{c^2k^2}{2}\big(\cosh(2\xi) - \cos(2\eta)\big)V = 0,$$

where $V(\xi, \eta) = U(x, y)$. Putting $V(\xi, \eta) = R(\xi)\Phi(\eta)$, we obtain the Mathieu equation

$$\frac{d^2\Phi}{d\eta^2} + (a - 2q\cos(2\eta))\Phi = 0$$

and the modified Mathieu equation

$$\frac{d^2 R}{d\xi^2} - (a - 2q\cosh(2\xi))R = 0,$$

where $a$ is the separation constant and the parameter $q = c^2k^2/4$. Note that the modified Mathieu equation can be transformed to the Mathieu equation by the mapping $\eta = \pm\sqrt{-1}\,\xi$.

Mathieu's equation is a linearized model of an inverted pendulum, where the pivot point oscillates periodically in the vertical direction (see [20]). It is also derived in the study of celestial mechanics (see [3,4]) and in the vibration of string whose tension is changed periodically (Melde's experiment). In fluid dynamics, we can find many examples of waves being described by Mathieu's equation. The research of Faraday surface waves is very active (see [2,6,9,21]). For other applications of Mathieu's equation, see McLachan [19].

The purpose of this paper is to give a simple parametric region that guarantees all nontrivial solutions of the generalized Mathieu equation (1.1) are oscillatory (respectively, nonoscillatory) (see Section 2 for the definitions).

Using the method mentioned by McLachan [19, p. 29], we can determine the boundary of the largest parametric oscillation region for Mathieu's equation (1.2). The boundary is described by the infinite continued fraction

$$\alpha = \frac{\beta^2}{2(4+\alpha)-}\,\frac{\beta^2}{2(16+\alpha)-}\,\frac{\beta^2}{2(36+\alpha)-}\,\frac{\beta^2}{2(64+\alpha)-}\cdots\frac{\beta^2}{2(4n^2+\alpha)-}\cdots, \tag{1.3}$$

where $n = 1, 2, \ldots$. To be precise, $\beta^2(\alpha)$ is the smallest positive root of the equation

$$\alpha = \frac{\lambda}{2(4+\alpha)-}\,\frac{\lambda}{2(16+\alpha)-}\,\frac{\lambda}{2(36+\alpha)-}\,\frac{\lambda}{2(64+\alpha)-}\cdots\frac{\lambda}{2(4n^2+\alpha)-}\cdots$$