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Extremes of $\alpha(t)$-locally stationary Gaussian processes with non-constant variances

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ABSTRACT

With motivation from [9], in this paper we derive the exact tail asymptotics of $\alpha(t)$-locally stationary Gaussian processes with non-constant variance functions. We show that some certain variance functions lead to qualitatively new results.

1. Introduction and main result

For $X(t),\ t \in [0,T],\ T > 0$ a centered stationary Gaussian process with unit variance and continuous sample paths Pickands derived in [20] that

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} \sim T\mathcal{H}_\alpha a^{1/\alpha} u^{2/\alpha} \mathbb{P}\left\{X(0) > u\right\}, \quad u \to \infty, \tag{1}$$

provided that the correlation function r satisfies

$$1 - r(t) \sim a\,|t|^\alpha, \quad t \downarrow 0, \quad a > 0, \quad \text{and } r(t) < 1,\ \forall\, t \neq 0, \tag{2}$$

with $\alpha \in (0,2]$ ($\sim$ means asymptotic equivalence when the argument tends to 0 or ∞). Here the classical Pickands constant $\mathcal{H}_\alpha$ is defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} T^{-1}\mathbb{E}\left\{\sup_{t\in[0,T]} e^{\sqrt{2}B_\alpha(t) - t^\alpha}\right\},$$

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where $B_\alpha(t), t \geq 0$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, see [20,21,5,12,10,14,7,23,11,13,6,15] for various properties of $\mathcal{H}_\alpha$.

The deep contribution [3] introduced the class of locally stationary Gaussian processes with index α, i.e., a centered Gaussian process $X(t), t \in [0, T]$ with a constant variance function, say equal to 1, and correlation function satisfying

$$r(t, t+h) = 1 - a(t)|h|^\alpha + o(|t|^\alpha), \ h \to 0,$$

uniformly with respect to $t \in [0, T]$, where $\alpha \in (0, 2]$ and $a(t)$ is a bounded, strictly positive and continuous function.

Clearly, the class of locally stationary Gaussian processes includes the stationary ones. It allows for some minor fluctuations of dependence at t and at the same time keeps stationary structure at the local scale. See [3,4,18] for studies on the locally stationary Gaussian processes with index α.

In [9] the tail asymptotics of the supremum of $\alpha(t)$-locally stationary Gaussian processes are investigated. Such processes and random fields are of interest in various applications, see [9] and the recent contributions [2,16,17]. Following the definition in [9], a centered Gaussian process $X(t), t \in [0, T]$ with continuous sample paths and unit variance is $\alpha(t)$-locally stationary if the correlation function $r(\cdot, \cdot)$ satisfies the following conditions:

(i) $\alpha(t) \in C([0, T])$ and $\alpha(t) \in (0, 2]$ for all $t \in [0, T]$;
(ii) $a(t) \in C([0, T])$ and $0 < \inf\{a(t) : t \in [0, T]\} \leq \sup\{a(t) : t \in [0, T]\} < \infty$;
(iii) uniformly for $t \in [0, T]$

$$1 - r(t, t+h) = a(t)|h|^{\alpha(t)} + o(|h|^{\alpha(t)}), \ h \to 0,$$

where $f(t) \in C(\mathcal{T})$ means that $f(t)$ is continuous on $\mathcal{T} \subset \mathbb{R}$.

In this paper, we shall consider the case that the variance function $\sigma^2(t) = Var(X(t))$ is not constant, assuming instead that:
(iv) $\sigma(t)$ attains its maximum equal to 1 over $[0, T]$ at the unique point $t_0 \in [0, T]$ and for some constants $c, \gamma > 0$,

$$\frac{1}{\sigma(t)} = 1 + ce^{-|t-t_0|^{-\gamma}}(1 + o(1)), \quad t \to t_0.$$

A crucial assumption in our result is that similar to the variance function, the function $\alpha(t)$ has a certain behavior around the extreme point t_0. Specifically, as in [9] we shall assume:
(v) there exist $\beta, \delta, \ b > 0$ such that

$$\alpha(t + t_0) = \alpha(t_0) + b|t|^\beta + o(|t|^{\beta+\delta}), \quad t \to 0.$$

Remark 1.1. We remark that t_0 does not need to be the unique point such that $\alpha(t)$ is minimal on $[0, T]$, which is different from [9]. For instance, $[0, T] = [0, 2\pi]$, $t_0 = 0$ and $\alpha(t) = 1 + \frac{1}{2}\sin(t)$, then 0 is not the minimum point of $\alpha(t)$ over $[0, 2\pi]$ which means assumptions about $\alpha(t)$ in [9] are not satisfied but assumption (v) here is satisfied with

$$\alpha(t) = 1 + \frac{1}{2}|t| + o(|t|^{\frac{3}{2}}), \ t \to 0.$$

Below we set $\alpha := \alpha(t_0)$, $a := a(t_0)$ and write Ψ for the survival function of an $N(0, 1)$ random variable. Further, define $0^a = \infty$ for $a < 0$. Our main result is stated in the next theorem.