J. Math. Anal. Appl. 446 (2017) 264–275

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An analogue to a result of Takahashi ☆

Dragana S. Cvetković-Ilić *, Vladimir Pavlović

Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia

ARTICLE INFO

Article history:
Received 30 September 2015
Available online 1 September 2016
Submitted by L. Fialkow

Keywords:
Operator matrix
Injectivity
Invertibility
Operator range

ABSTRACT

In this paper we completely answer the following question: for given $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $C \in \mathcal{B}(\mathcal{L},\mathcal{K})$ does there exist $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that $A + CX$ is injective? As an application of the obtained results, for given operators $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, separate necessary conditions and sufficient conditions on the triple (A, B, C) for the existence of an operator X such that the 2×2 operator matrix $M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix}$ is injective are presented.

1. Motivation

Let $\mathcal{H},\mathcal{K}$ be separable Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H},\mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. If $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ are fixed, by M_X we denote the operator in $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ given by

$$M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

which depends on $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$.

Completion of partially given operator matrices to operators of fixed prescribed type is an extensively studied area of operator theory, which is a topic of many various currently undergoing investigations.

As for completions of the operator matrices of the above given form M_X, the first to ever address any kind of questions (for separable Hilbert spaces not necessarily of finite dimension) related to it was Takahashi. More specifically, in his paper [14] he gave necessary and sufficient conditions for the existence of $X \in \mathcal{B}(\mathcal{H})$ such that M_X is invertible.

☆ The work of the first author is supported by Grant No. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia and the work of the second author is supported by Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

* Corresponding author.
E-mail addresses: dragana@pmf.ni.ac.rs (D.S. Cvetković-Ilić), vlada@pmf.ni.ac.rs (V. Pavlović).

http://dx.doi.org/10.1016/j.jmaa.2016.08.060

The key result obtained in [14] that allowed for him to completely solve the problem of completion of M_X to invertibility was the one that characterizes the pairs of operators (A, C) for which the operator $A+CX$ is invertible for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, which is a result also of interest in connection with the spectrum assignment problem in systems theory.

Although Takahashi's paper was published in '95 there have only been several papers since, namely [3, 8,9,13,17,18], which deal with various completions of the operator matrix of the form M_X. Actually in [17] exactly the same problem as in [14] was considered but using methods of geometrical structure of operators and in it some necessary and sufficient conditions were given different than those from [14]. Of all the cited papers the only one that addresses completion to injectivity is [3], and it does it under the special assumption that the operator C is with closed range. So, the problem of completing the operator matrix M_X to injectivity is still open.

In attempt to answer this open question, we have run into a similar problem as Takahashi did. More precisely, instead of the question of existence of an operator X for which $A+CX$ is invertible, we have faced ourselves with the following one: given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ does there exist an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $A + CX$ is injective?

In this paper we completely answer the above question. As an application of the obtained results we partially solve the problem of completion of the operator matrix M_X to injectivity.

2. Notation and preliminaries

All Hilbert spaces under consideration are assumed to be separable.

For subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{H}$ with $\mathcal{X} \subseteq \mathcal{Y}$, we set $codim_{\mathcal{Y}}\,\mathcal{X} = dim\,\mathcal{Y}/\mathcal{X}$ and, if $\mathcal{X}$ is closed, use the symbol $P_{\mathcal{X}}$ to denote the orthogonal projection onto $\mathcal{X}$. For a set M by 1_M we denote the identity function on M; we write simply 1 if M is understood. For a given operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. We use the standard notation $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \mathrm{codim}\mathcal{R}(A)$ and $d(A) = \dim \mathcal{R}(A)^{\perp}$. As usual, $\sigma(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not invertible}\}$ is the spectrum of A, $\sigma_p(A) = \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda - A) \neq \{0\}\}$ is the point spectrum of A and $\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}$ is the essential spectrum of A. Analogously, left (right) spectrum and left (right) essential spectrum of A can be defined.

If $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $\mathcal{M}$ is a subspace of $\mathcal{K}$ then the restriction of operator A to the subspace $\mathcal{M}$ will be denoted by $A|_{\mathcal{M}}$.

By an *operator range* we shall mean a subspace $\mathcal{K} \subseteq \mathcal{H}$ of a separable Hilbert space $\mathcal{H}$ such that $\mathcal{R}(A) = \mathcal{K}$ for some separable Hilbert space $\mathcal{H}_0$ and some $A \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$.

In the lemma below we collect a few basic facts about operator ranges that will be used in the paper without any explicit mention.

Lemma 2.1. (i) *If $L \subseteq \mathcal{H}$ is an operator range and $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$, then $A[L]$ is an operator range as well.*

(ii) *If $L_1, L_2 \subseteq \mathcal{H}$ are operator ranges then $L_1 + L_2$ is an operator range as well.*

(iii) *If $L_1, L_2 \subseteq \mathcal{H}$ are operator ranges then $L_1 \cap L_2$ is an operator range as well.*

(iv) *If $L \subseteq \mathcal{H}$ is an operator range and $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$, then $A^{-1}[L]$ is an operator range as well.*

Proof. (i) is immediate from the definition and (ii) and (iii) are proved e.g. in [4].

To prove (iv) let $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$ be such that $L = \mathcal{R}(B)$. Consider the operators $A_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \oplus \mathcal{H})$ and $B_1 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H})$ given by

$$A_1 = \begin{bmatrix} 1 \\ A \end{bmatrix} : \mathcal{H}_1 \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H} \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H} \end{bmatrix}$$

and let $P_1 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}, \mathcal{H}_1)$ be the orthogonal projection onto $\mathcal{H}_1$. From