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Tauberian convolution operators acting on $L_1(G)$ ☆

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ABSTRACT

We study the convolution operators T_μ which are tauberian as operators acting on the group algebras $L_1(G)$, where G is a locally compact abelian group and μ is a complex Borel measure on G. We show that these operators are invertible when G is non-compact, and that they are Fredholm when they have closed range and G is compact. In the remaining case, when G is compact and $R(T_\mu)$ is not assumed to be closed, we prove that T_μ is Fredholm when the singular continuous part of μ with respect to the Haar measure on G is zero.

1. Introduction

Tauberian operators, introduced by Kalton and Wilansky in [8], have found many applications in Banach space theory (see Chapter 5 of [5]). The tauberian operators from $L_1(\mu)$ into a Banach space were studied in [4], where several characterizations in terms of disjoint sequences were obtained, and the problem whether all tauberian operators $T : L_1(\mu) \to L_1(\mu)$ are upper semi-Fredholm (have closed range and finite dimensional kernel) was raised. A counterexample to this problem was obtained in [7]: there exists a tauberian operator acting on $L_1(0, 1)$ with non-closed range, but it is not known if there exists a counterexample with closed range.

Here we consider the mentioned problem for convolution operators T_μ acting on the Banach algebra $L_1(G)$, where G is a locally compact abelian (LCA for short) group. We show that the tauberian operators T_μ are invertible when the group G is non-compact, and that they are Fredholm when G is compact and they have closed range or the singular continuous part (with respect to the Haar measure on G) of the associated measure μ is zero. These results provide new characterizations of the Fredholm multipliers of the group algebras $L_1(G)$ described in [1, Theorem 5.97].

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Throughout the paper X and Y are (complex) Banach spaces, and G is a locally compact abelian group. We consider (continuous linear) operators $T : X \to Y$, and we denote by $R(T)$ and $N(T)$ the range and the kernel of T. An operator $T : X \to Y$ is *tauberian* if the second conjugate $T^{**} : X^{**} \to Y^{**}$ satisfies $T^{**-1}(Y) = X$, and the operator T is *weakly compact* if $T^{**}(X^{**}) \subset Y$. The properties of being tauberian and being weakly compact are "opposite", and given $T : X \to Y$ tauberian and $K : X \to Y$ weakly compact, $T + K$ tauberian.

An operator $T : X \to Y$ is *upper semi-Fredholm* if it has closed range and finite dimensional kernel, and T is *Fredholm* if it has closed finite codimensional range and finite dimensional kernel. Upper semi-Fredholm operators are tauberian [5, Theorem 2.1.5], and from our point of view can be considered as "trivial" tauberian operators.

We denote by $G = (G, +, 0)$ an infinite LCA group, and by m the Haar measure on G. Moreover $L_1(G)$ is the space of m-integrable complex functions on G endowed with the L_1-norm $\|\cdot\|_1$, and $M(G)$ denotes the space of complex Borel measures on G endowed with the variation norm. Note that $L_1(G)$ can be identified with the subspace of those $\mu \in M(G)$ that are absolutely continuous with respect to m by associating to $f \in L_1(G)$ the Borel measure m_f on G defined by $m_f(A) = \int_A f(x)dm(x)$. The space $L_1(G)$ with the convolution $(f \star g)(x) = \int_G f(x-y)g(y)dm(y)$ is a commutative Banach algebra.

Given $\mu \in M(G)$ and $f \in L_1(G)$ the expression $(\mu \star f)(x) = \int_G f(x-y)d\mu(y)$ defines $\mu \star f \in L_1(G)$ satisfying $\|\mu \star f\|_1 \leq \|\mu\| \cdot \|f\|_1$. Thus for every $\mu \in M(G)$ we obtain a *convolution operator* T_μ on $L_1(G)$ defined by $T_\mu f = \mu \star f$, and satisfying $\|T_\mu\| = \|\mu\|$. Moreover, given $\mu, \nu \in M(G)$, the convolution of measures $\mu \star \nu \in M(G)$ is commutative [11]. Therefore $T_{\mu\star\nu} = T_\mu T_\nu = T_\nu T_\mu$.

For each $r \in G$ the *translation operator* T_r on $L_1(G)$, defined by $(T_r f)(x) = f(x - r)$, is a bijective isometry on $L_1(G)$. The convolution operators can be characterized as those operators $T : L_1(G) \to L_1(G)$ that commute with translations ($T_r T = T T_r$ for each $r \in G$) [9, Theorem 0.1.1]. Note that T_r is the convolution operator associated to the unit measure δ_r concentrated at $\{r\}$.

Let Γ denote the dual group of G. Given $f \in L_1(G)$ and $\mu \in M(G)$, the Fourier transform $\widehat{f} : \Gamma \to \mathbb{C}$ of f and the Fourier–Sieltjes transform $\widehat{\mu} : \Gamma \to \mathbb{C}$ of μ are defined by $\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dm(x)$ and $\widehat{\mu}(\gamma) = \int_G \gamma(-x)d\mu(x)$.

For basic results on Fredholm theory, tauberian operators and Fourier analysis we refer to [1,5] and [11].

2. Preliminary results

Given $f \in L_1(\mu)$, we denote $\mathrm{supp}(f) = \{t : f(t) \neq 0\}$. We say that a sequence (f_n) in $L_1(\mu)$ is *disjoint* if $\mu\big(\mathrm{supp}(f_k) \cap \mathrm{supp}(f_l)\big) = 0$ for $k \neq l$.

The following result was proved in [4] (see also [5, Chapter 4]) when μ is a non-atomic finite measure, but the arguments given there are valid when μ is σ-finite.

Theorem 2.1. *[4, Theorems 2 and 6] Let μ be a σ-finite measure. For an operator $T : L_1(\mu) \to Y$ the following assertions are equivalent:*

(1) *T is tauberian;*
(2) $\liminf_{n\to\infty} \|Tf_n\| > 0$ *for every disjoint normalized sequence (f_n) in $L_1(\mu)$;*
(3) *there exists a number $r > 0$ such that* $\liminf_{n\to\infty} \|Tf_n\| > r$ *for every disjoint normalized sequence (f_n) in $L_1(\mu)$;*
(4) $\liminf_{n\to\infty} \|Tf_n\| > 0$ *for every normalized sequence (f_n) in $L_1(\mu)$ satisfying* $\lim_{n\to\infty} \mu(\mathrm{supp}(f_n)) = 0$.

The structure of convolution operators with closed range was described by Host and Parreau.

Theorem 2.2. *[6, Théorème 1] Let G be a LCA group and let $\mu \in M(G)$. Then the convolution operator T_μ has closed range if and only if $\mu = \nu \star \xi$, where $\nu, \xi \in M(G)$, ν is invertible and ξ is idempotent.*