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# On rational matrix exact covering systems of $\mathbb{Z}^n$ and its applications to Ramanujan's forty identities

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ABSTRACT

For a nonsingular integer matrix $B$, the set of cosets of the quotient module $\mathbb{Z}^n/B\mathbb{Z}^n$ forms an exact covering system (ECS) of $\mathbb{Z}^n$. In this paper, we use the Smith normal form to obtain another type of matrix ECS with rational entries which we call rational matrix ECS. Using rational matrix ECS of $\mathbb{Z}^2$, we prove eight identities in Ramanujan's list of forty identities for the Rogers–Ramanujan functions, as well as some other identities.



## 1. Matrix ECS

A set of congruences $a_i \pmod{n_i}$ with $1 \le i \le k, i \in \mathbb{Z}$ is called a covering system of $\mathbb{Z}$ if every integer $y$ satisfies $y \equiv a_i \pmod{n_i}$ for at least one value of $i$. A covering system in which each integer is covered by only one congruence is called an exact covering system (ECS).

We can extend the idea of ECS to $\mathbb{Z}^n$. Specifically, we focus on the structure of the quotient module $\mathbb{Z}^n/B\mathbb{Z}^n$, for a nonsingular $n \times n$ integer matrix $B$. For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, we define $\mathbf{a} \equiv \mathbf{b} \pmod{B}$ if and only if $\mathbf{a} - \mathbf{b} \in B\mathbb{Z}^n$. Using the Smith normal form, in [6], M. Fiol proved the following result.

**Theorem 1.1.** *For a given nonsingular $n \times n$ matrix $B$, the quotient group*

$$\mathbb{Z}^n/B\mathbb{Z}^n \simeq \mathbb{Z}/s_1\mathbb{Z} \times \mathbb{Z}/s_2\mathbb{Z} \times \cdots \times \mathbb{Z}/s_n\mathbb{Z},$$

*where $s_1|s_2|\cdots|s_n$ are the invariant factors of matrix $B$. Therefore, $|\mathbb{Z}^n/B\mathbb{Z}^n| = s_1s_2\cdots s_n = |\det B|$. Moreover, $\mathbb{Z}^n/B\mathbb{Z}^n$ is cyclic if and only if $s_{n-1} = 1$, or equivalently $s_n = |\det B|$.*

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The quotient group $\mathbb{Z}^n/B\mathbb{Z}^n$ has exactly $|\det B|$ elements. Let $|\det B| = k$. Taking coset representatives $c_i$ from each equivalent class, we obtain a matrix ECS of $\mathbb{Z}^n$ as

$$\bigcup_{i=0}^{k-1}\{B\mathbb{Z}^n + c_i\}.$$

The matrix ECS can be used to regroup multiple summations. Let $f(\mathbf{x})$ be a complex-valued multivariable function. Provided that the summation converges absolutely, the equality

$$\sum_{\mathbf{x}\in\mathbb{Z}^n} f(\mathbf{x}) = \sum_{i=0}^{k-1}\sum_{\mathbf{y}\in\mathbb{Z}^n} f(B\mathbf{y} + c_i) \tag{1.1}$$

holds.

We are more interested in ECS with simple structures. A type of ECS with nice coset representatives is given by

$$\{B\mathbb{Z}^n + ie_j\}, i = 0, 1, \cdots, k-1, \tag{1.2}$$

where $e_j = (0, 0, \cdots, 0, 1, 0, 0, \cdots, 0)^T$ is the $j$th elementary column vector in $\mathbb{Z}^n$. We call (1.2) *simple ECS*. We call the matrix $B$ *simple covering matrix*. In [4], Cao studied some simple ECS and used them to prove some identities for theta functions. Here we give a sufficient and necessary condition for simple ECS.

**Theorem 1.2.** *A nonsingular integer matrix $B$ is a simple covering matrix if and only if there exists $1 \le j \le n$, such that the elements of $B_j^*$ are coprime, where $B_j^*$ is the $j$th column of $B^*$, the adjugate of $B$. In this case, $\{B\mathbb{Z}^n + ie_j\}$ is an ECS of $\mathbb{Z}^n$, where $i$ runs through any complete residue system modulo $|\det B|$.*

**Proof.** Note that $\mathbb{Z}^n/B\mathbb{Z}^n$ is a simple ECS if and only if, for some fixed $j$, the equation

$$Bx = re_j$$

has no integer solution for all $r = 1, 2, \cdots, k-1$. We prove it by contraposition.

Solving the above equation, we have

$$x = \frac{B^*}{\det B} re_j = \frac{r}{k} B_j^*. \tag{1.3}$$

Set $\gcd(B_{1j}^*, B_{2j}^*, \cdots, B_{nj}^*) = d$. Note that $d|k$. If $d > 1$, then (1.3) has integer solutions when $r = k/d$. On the other hand, if (1.3) has integer solutions for some $1 \le r < k$, then $k/(r,k)$ divides all elements of $B_j^*$. Therefore $d \ge k/(r,k) > 1$.

Therefore we see that (1.3) has no integer solutions if and only if the elements of $B_j^*$ are coprime. □

**Corollary 1.1.** *A $2 \times 2$ integer matrix $B$ is a simple covering matrix if and only if $gcd(b_{11}, b_{12}) = 1$ or $gcd(b_{21}, b_{22}) = 1$.*

**Proof.** This is a special case of Theorem 1.2 when $B$ is a $2 \times 2$ matrix. □

Next we introduce a more sophisticated structure than integer matrix ECS.