J. Math. Anal. Appl. 446 (2017) 345–364

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Large values of L-functions from the Selberg class

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ARTICLE INFO

Article history:
Received 18 March 2016
Available online 5 September 2016
Submitted by B.C. Berndt

Keywords:
Selberg class
L-function
Large values
Resonance method
Diophantine approximation

ABSTRACT

In the present paper we prove lower bounds for L-functions from the Selberg class, by this means improving earlier results obtained by the second author together with Jörn Steuding. We formulate two theorems which use slightly different technical assumptions, and give two totally different proofs. The first proof uses the "resonance method", which was introduced by Soundararajan, while the second proof uses methods from Diophantine approximation which resemble those used by Montgomery. Interestingly, both methods lead to roughly the same lower bounds, which fall short of those known for the Riemann zeta function and seem to be difficult to be improved. Additionally to these results, we also prove upper bounds for L-functions in the Selberg class and present a further application of a theorem of Chen which is used in the Diophantine approximation method mentioned above.

1. Introduction

It is well known that the absolute value of the Riemann zeta function $\zeta(\sigma + it)$ takes arbitrarily large and arbitrarily small values when t runs through the real numbers and $\sigma \in [1/2, 1)$ is a fixed real number. However, the growth of the Riemann zeta function as a function of t (for fixed σ) cannot be too fast, since its absolute value is bounded by a power of t. More precisely, if $\mu_\zeta(\sigma)$ denotes the infimum over all $c \geq 0$ satisfying $\zeta(\sigma + it) \ll t^c$ for sufficiently large t, then one can show that $\mu_\zeta(\sigma) \leq (1 - \sigma)/2$ for $0 \leq \sigma \leq 1$. Although the upper bound for $\mu_\zeta(\sigma)$ has been improved by many mathematicians, especially for $\sigma = 1/2$, it is yet unproved (but widely believed) that $\mu_\zeta(\sigma) = 0$ for $\sigma \geq 1/2$ (for more details we refer to [10] or [20]). As evidence for the truth of this conjecture one can regard the Riemann hypothesis, which implies that

$$\log \zeta(\sigma + it) \ll \frac{(\log t)^{2-2\sigma}}{\log \log t}, \qquad \text{for} \quad \frac{1}{2} \leq \sigma < 1.$$

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http://dx.doi.org/10.1016/j.jmaa.2016.08.044

Therefore, it is natural to ask for omega results on $\zeta(\sigma+it)$. The first answer was given by Titchmarsh (see [20, Theorem 8.12]), who proved that for any $\sigma \in [1/2,1)$ and every $\varepsilon > 0$ the inequality $|\zeta(\sigma+it)| > \exp\left((\log t)^{1-\sigma-\varepsilon}\right)$ holds for arbitrarily large values of t. In 1977, Montgomery [12] improved this result for $\sigma \in (1/2,1)$ by proving that for any fixed $\sigma \in (1/2,1)$ and every sufficiently large T there exists t such that $T^{(\sigma-1/2)/3} \le t \le T$ and

$$|\zeta(\sigma+it)| \ge \exp\left(\frac{1}{20}\left(\sigma-\frac{1}{2}\right)^{1/2}\frac{(\log T)^{1-\sigma}}{(\log\log T)^{\sigma}}\right). \tag{1}$$

Moreover, he showed that under the Riemann Hypothesis the above inequality can be extended to $\sigma \in [1/2,1)$ with a slightly better constant and better range of t.

The first unconditional proof of Montgomery's theorem for $\sigma = 1/2$ was given by Balasubramanian and Ramachandra [4]. The best result currently known is due to Bondarenko and Seip [5], who very recently achieved a breakthrough by proving that

$$\max_{T^{1/2}\le t\le T}\left|\zeta\left(\frac{1}{2}+it\right)\right| \ge \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\frac{\log T\log\log\log T}{\log\log T}}\right).$$

Their proof is based on the so-called resonance method, which was introduced by Soundararajan [17], and on a connection between extreme values of the Riemann zeta function and certain sums involving greatest common divisors (GCD sums). This connection was discovered by Hilberdink [9]. Recently, the first author [1] succeeded in applying the resonance method with an extremely long resonator such that he could recapture Montgomery's results by the resonance method, off the critical line, an idea which also plays a crucial role in the omega result of Bondarenko and Seip.

Similar problems of finding extreme values were also investigated for other zeta and L-functions, and it was shown that Montgomery's approach can be applied to some generalizations of the Riemann zeta function. For example, Balakrishnan [3] showed that Dedekind zeta functions take large values of order $\exp(c(\log T)^{1-\sigma}/(\log\log T)^{\sigma})$, and Sankaranarayanan and Sengupta [16] generalized Montgomery's theorem to a wide class of L-functions defined by Dirichlet series with real coefficients under some natural analytic and arithmetic conditions.

Recently, the second author and Steuding [15] investigated further refinements of Montgomery's reasoning and proved that for every L-function $L(s) = \sum_{n\ge1} a_L(n)n^{-s}$ from the Selberg class which satisfies $L(s) \ne 0$ for $\sigma > 1/2$ we have

$$\max_{t\in[T,2T]}|L(\sigma+it)| \ge \exp\left(c\frac{(\log T)^{1-\sigma}}{(\log\log T)^{2-\sigma}}\right) \tag{2}$$

for some explicitly given constant $c > 0$ and sufficiently large T, under the additional assumption that the coefficients of L satisfy a prime number theorem with remainder term in the form

$$\sum_{p\le x}|a_L(p)| = \kappa\frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right), \qquad (\kappa > 0). \tag{3}$$

Note that Montgomery's argument requires a prime number theorem in order to get a lower bound for the sum of $|a_L(p)|$ over primes in some interval, which might be estimated from below by the sum of $|a_L(p)|^2$, provided $|a_L(p)| \ll 1$. Hence, the condition (3) can be replaced by the more natural assumption that L has a polynomial Euler product and satisfies the Selberg normality conjecture in the stronger form

$$\sum_{p\le x}|a_L(p)|^2 = \kappa\frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right), \qquad (\kappa > 0). \tag{4}$$