J. Math. Anal. Appl. 446 (2017) 365–382

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lusin theorem, GLT sequences and matrix computations: An application to the spectral analysis of PDE discretization matrices

Carlo Garoni [a,b,*], Carla Manni [a], Stefano Serra-Capizzano [b,c], Debora Sesana [b], Hendrik Speleers [a]

[a] *Department of Mathematics, University of Rome 'Tor Vergata', Via della Ricerca Scientifica 1, 00133 Rome, Italy*
[b] *Department of Science and High Technology, University of Insubria, Via Valleggio 11, 22100 Como, Italy*
[c] *Department of Information Technology, Uppsala University, Box 337, SE-751 05 Uppsala, Sweden*

A R T I C L E I N F O

Article history:
Received 30 November 2015
Available online 24 August 2016
Submitted by W.L. Wendland

Keywords:
Lusin theorem
Generalized locally Toeplitz sequences
Spectral distribution and symbol
PDE discretization matrices
Isogeometric analysis

A B S T R A C T

In this paper we consider a general d-dimensional second-order elliptic Partial Differential Equation (PDE) with variable coefficients, and we extend previous results on the spectral distribution of discretization matrices arising from B-spline Isogeometric Analysis (IgA). First, we provide the spectral symbol of the Galerkin B-spline IgA stiffness matrices, under the assumption that the PDE coefficients only belong to L^∞. This symbol describes the asymptotic spectral distribution when the fineness parameters tend to zero (so that the matrix-size tends to infinity). Second, we prove the positive semi-definiteness of the $d \times d$ symmetric matrix in the Fourier variables $(\theta_1, \ldots, \theta_d)$, which appears in the expression of the symbol. This matrix is related to the discretization of the (negative) Hessian operator, and its positive semi-definiteness implies the non-negativity of the symbol. The mathematical arguments used in our derivation are based on the Lusin theorem, on the theory of Generalized Locally Toeplitz (GLT) sequences, and on careful Linear Algebra manipulations of matrix determinants. These arguments are very general and can also be applied to other (local) PDE discretization methods, different from B-spline IgA.

1. Introduction

Consider the following second-order elliptic Partial Differential Equation (PDE) with variable coefficients and homogeneous Dirichlet boundary conditions:

* Corresponding author at: Department of Science and High Technology, University of Insubria, Via Valleggio 11, 22100 Como, Italy.

E-mail addresses: garoni@mat.uniroma2.it, carlo.garoni@uninsubria.it (C. Garoni), manni@mat.uniroma2.it (C. Manni), stefano.serrac@uninsubria.it, stefano.serra@it.uu.se (S. Serra-Capizzano), debora.sesana@uninsubria.it (D. Sesana), speleers@mat.uniroma2.it (H. Speleers).

http://dx.doi.org/10.1016/j.jmaa.2016.08.040
0022-247X/

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\alpha} \cdot \nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded open domain with Lipschitz boundary, $K : \Omega \to \mathbb{R}^{d\times d}$ is a Symmetric Positive Definite (SPD) matrix of functions in $L^\infty(\Omega)$, $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^d$ is a vector of functions in $L^\infty(\Omega)$, $\gamma \in L^\infty(\Omega)$, $\gamma \geq 0$ and $\mathrm{f} \in L^2(\Omega)$. Any linear discretization of the given differential problem for some sequence of stepsizes h tending to zero leads to a sequence of linear systems $A_h\mathbf{u} = \mathbf{f}$, where the size of A_h tends to ∞ as $h \to 0$. For the numerical solution of such linear systems, it is important to understand the spectral properties of the matrices A_h. The spectral distribution of a sequence of matrices is a relevant concept. Roughly speaking, if the sequence of matrices $\{A_h\}$ has a spectral distribution described by the function f (and f is at least Riemann-integrable), then the eigenvalues of A_h behave like a sampling of f over an equispaced grid on the domain of f. In this case, the function f is called the (spectral) symbol of the sequence $\{A_h\}$.

The spectral distributions of sequences of discretization matrices arising from B-spline Isogeometric Analysis (IgA) approximations of (1.1) were investigated in a series of recent papers. In particular, the Galerkin B-spline IgA discretization was addressed in [9–11]. The symbol of the corresponding sequence of stiffness matrices was computed and studied in [9,10] in the simplified constant-coefficient case, where K is the identity matrix and $\Omega = (0, 1)^d$. Afterwards, in [11], the spectral study was generalized to the case where the PDE coefficients are continuous functions and Ω is an arbitrary domain described by a suitable geometry map $\mathbf{G} : [0, 1]^d \to \overline{\Omega}$. A similar spectral study was performed in [8] for the sequence of collocation matrices arising in the B-spline IgA collocation context. In addition, the information contained in the symbol was successfully exploited in the design/analysis of optimal multigrid methods for the numerical solution of the linear systems involved in IgA discretizations [5–7]. Especially for high values of the B-spline degrees $\boldsymbol{p} := (p_1, \ldots, p_d)$, some problems with classical iterative methods were detected and a multi-iterative multigrid strategy to solve them was described in [5–7].

In this paper, we extend the above mentioned results about the spectral distribution and the related symbol in the following two ways.

- We identify the complete symbol f describing the spectral distribution of the sequence of stiffness matrices related to the Galerkin B-spline IgA discretization of (1.1). We only require that the PDE coefficients belong to $L^\infty(\Omega)$. The argument used in our derivation is based on two main tools:
  - the Lusin theorem [15], to approximate a measurable function by a continuous function;
  - the theory of Generalized Locally Toeplitz (GLT) sequences [12,18,19], which stems from Tilli's work [20] and from the theory of classical Toeplitz operators (see, e.g., [2]).
- We prove the positive semi-definiteness of the $d \times d$ symmetric matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ in the Fourier variables $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_d)$, which appears in the expression of the symbol f. The matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ is related to the discretization of the (negative) Hessian operator, and is sometimes referred to as 'the symbol of the (negative) Hessian operator'. The positive semi-definiteness of $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ implies the non-negativity of f and is an important property for the spectral distribution of discretization matrices. Moreover, we show that this result straightforwardly extends to the B-spline IgA collocation case.

The remainder of the paper is organized as follows. In Section 2 we briefly describe the Galerkin IgA approximation based on uniform tensor-product B-splines of degrees $\boldsymbol{p}$ of the full elliptic problem (1.1), and we present our two main spectral results: the identification of the complete symbol f describing the spectral distribution of the corresponding stiffness matrices, and the property of positive semi-definiteness for the matrix $H_{\boldsymbol{p}}(\boldsymbol{\theta})$ which appears in the expression of the symbol f. Some preliminaries on GLT sequences are collected in Section 3. Then, we prove our two main results in Sections 4 and 5, respectively. We extend