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Boundedness in a Keller–Segel system with external signal production

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We study the Neumann initial-boundary problem for the chemotaxis system

$$\begin{cases} u_t &= \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ \ t>0,\\ v_t &= \Delta v - v + u + f(x,t), & x\in\Omega,\ \ t>0,\\ \frac{\partial u}{\partial\nu} &= \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) &= u_0(x),\ v(x,0)=v_0(x), & x\in\Omega \end{cases}$$

in a smooth, bounded domain $\Omega\subset\mathbb{R}^n$ with $n\geq 2$ and $f\in \mathrm{L}^\infty([0,\infty);\mathrm{L}^{\frac{n}{2}+\delta_0}(\Omega))\cap C^\alpha(\Omega\times(0,\infty))$ with some $\alpha>0$ and $\delta_0\in(0,1)$. First we prove local existence of classical solutions for reasonably regular initial values. Afterwards we show that in the case of $n=2$ and f being constant in time, requiring the nonnegative initial data u_0 to fulfill the property $\int_\Omega u_0\,\mathrm{d}x<4\pi$ ensures that the solution is global and remains bounded uniformly in time. Thereby we extend the well known critical mass result by Nagai, Senba and Yoshida for the classical Keller–Segel model (coinciding with $f\equiv 0$ in the system above) to the case $f\not\equiv 0$. Under certain smallness conditions imposed on the initial data and f we furthermore show that for more general space dimension $n\geq 2$ and f not necessarily constant in time, the solutions are also global and remain bounded uniformly in time. Accordingly we extend a known result given by Winkler for the classical Keller–Segel system to the present situation.

1. Introduction

In mathematical biology, PDE systems of the form

$$\begin{cases} u_t &= \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ \ t>0,\\ v_t &= \Delta v - v + u, & x\in\Omega,\ \ t>0,\\ \frac{\partial u}{\partial\nu} &= \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) &= u_0(x),\ v(x,0)=v_0(x), & x\in\Omega \end{cases} \tag{KS}$$

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are widely being used to model the process of chemotaxis – a biological phenomenon of oriented movement of cells in response to some kind of chemical substance. Systems of this type were introduced in 1970, when Keller and Segel proposed a mathematical model describing the aggregation of some types of bacteria (see [13] and [14]). The model given above is a special case of the system originally stated in the pioneering works. Therein $u(x,t)$ represents the density of the cells and $v(x,t)$ denotes the concentration of an attracting chemical substance at place x and time t. The first equation of (KS) models the movement of the cells. This movement, while diffusive, also favors the direction toward higher concentration of the chemical substance. The second equation in (KS) models the assumptions that the chemical, while diffusing and degrading, is also consistently produced by the living cells themselves.

Similar variants to the system above have been used in modeling a wide array of biological phenomena, e.g. pattern formation in E. coli colonies [1] and cancer invasion of tissue [22] to just name a few. For a broader variety and further impressions on the biological background of this type we refer to the survey articles [2,10] and [11]. Often the occurrence of self-organized patterns such as aggregation, is identified with the blow-up of the solution, i.e. the existence of some $T \in (0,\infty]$ such that $\limsup_{t \nearrow T} \|u\|_{L^\infty(\Omega)} = \infty$. Accordingly, mathematical efforts are often focused on detecting unbounded solutions with finite time or infinite time blow up, or the lack of unbounded solutions. More generally one is interested in finding conditions on the initial data which either ensure or dismiss the existence of blow-up solutions all together.

The system (KS) has been thoroughly studied with regard to the boundedness of solutions. We briefly summarize some known results, where, if not stated otherwise, $\Omega \subset \mathbb{R}^n$ is an arbitrary, smooth and bounded domain and the initial values fulfill $u_0 \in C^0(\overline{\Omega})$, $v_0 \in C^1(\overline{\Omega})$ and are nonnegative. We denote by (u,v) the corresponding maximally extended classical solution of (KS):

If $n = 1$: Then (u,v) is global and bounded with regard to the $L^\infty(\Omega)$-norm [20].

If $n = 2$: If $\int_\Omega u_0 \, dx < 4\pi$ (or 8π in the radial symmetric setting), then (u,v) is global and bounded with regard to the $L^\infty(\Omega)$-norm [19,8].

On the other hand, for any $m > 4\pi$ with $m \notin \{4k\pi | k \in \mathbb{N}\}$ there exist initial data u_0, v_0 satisfying $\int_\Omega u_0 \, dx = m$ such that the respective solution blows up in finite or infinite time [12,21].

An additional important result is due to [17], where it was shown – in the radially symmetric setting – that finite time blow up is a quite typical occurrence, in the sense that for each $p \in (0,1)$, $q \in (1,2)$ and a given $T > 0$ the set of initial values leading to blow up before time T is dense in $\{(u_0,v_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) |$ radially symmetric and positive in $\overline{\Omega}\}$ with respect the topology in $L^p(\Omega) \times W^{1,q}(\Omega)$.

If $n \geq 3$: It was proven in [27] that there exists a bound for u_0 in $L^q(\Omega)$ and for ∇v_0 in $L^p(\Omega)$, with $q > \frac{n}{2}$ and $p > n$ such that the solution (u,v) is global in time and bounded. This result has further been extended to the critical case $q = \frac{n}{2}$ and $p = n$ [4]. Regarding blow up of solutions it was shown in [27] – in the radially symmetric setting – that for any $m > 0$ one can find initial data u_0 and v_0 satisfying $\int_\Omega u_0 \, dx = m$, such that the solution blows up in finite or infinite time. Furthermore, it was proven in [28] that – similar to the two dimensional case – blow up is also quite typical in space dimension three and above.

Similar results regarding the boundedness for the associated Cauchy problems on the whole space $\mathbb{R}^n$ have also been proven. See [6] for instance, where dimensions $n \geq 3$ are studied, or [16] for dimension $n = 2$.

The main purpose of this work is to examine if corresponding statements hold when an *additional external production of the signal chemical* is introduced to the system. More precisely, we shall study the initial-Neumann boundary value problem