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Some qualitative questions on the equation $-div(a(x, u, \nabla u)) = f(x, u)$

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ABSTRACT

In this article, we establish several applications of Picone's identity for the operator of the form

$$-div(a(x, u, \nabla u)),$$

such as Hardy type inequality, Sturmian comparison theorem, monotonicity property of the first eigenvalue, nonexistence of positive supersolutions and Caccioppoli inequality under certain conditions on a.

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1. Introduction

The classical Picone's identity says that for differentiable functions $v > 0$ and $u \geq 0$,

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0. \tag{1.1}$$

(1.1) has enormous applications to second-order elliptic equations and systems, see for instance [10–12,38] and the references therein. For a nonlinear version of (1.1), we refer to [45]. In order to apply (1.1) to equations involving p-Laplace operator, biharmonic operator and other general divergence type operators, (1.1) has been extended in several directions, see [13,21,27,32,33,35] and the references cited therein. W. Allegretto and Y.X. Huang [13] proved some qualitative results using the Picone's identity. J. Jaroš established the Picone's identity for Finsler p-Laplacian [33] and A-harmonic operator [32]. He also proved various qualitative results such as Caccioppoli type estimates, nonexistence of positive supersolutions, uniqueness and simplicity of the first eigenvalue, domain monotonicity property of the first eigenvalue, Barta-type inequality etc. B. Abdellaoui and I. Peral [1] used classical Picone's identity for p-Laplace operator to establish the Picone's inequality in integral form for $W^{1,p}(\Omega)$ functions. They used Picone's inequality to prove several results, see for instance [2,4–7,30,39,40]. Picone's identity for the operator

$$-div(a(x, \nabla u)),$$

where

$$a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfies certain conditions, is established by Kawohl et al. [35]. They proved that for differentiable functions $v > 0$ and $u \geq 0$, the following equality holds:

$$\begin{aligned}\langle a(x, \nabla u), \nabla u\rangle - \left\langle a(x, \nabla v), \nabla\left(\frac{u^p}{v^{p-1}}\right)\right\rangle &= \langle a(x, \nabla u), \nabla u\rangle - p\left\langle a\left(x, \frac{u}{v}\nabla v\right), \nabla u\right\rangle \\ &\quad + (p-1)\left\langle a\left(x, \frac{u}{v}\nabla v\right), \frac{u}{v}\nabla v\right\rangle,\end{aligned} \tag{1.2}$$

where $\langle \cdot\,, \cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$.
In the case when

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \tag{1.3}$$

and satisfies hypotheses given in Section 2, (1.2) can be obtained in the following form:

$$\begin{aligned}\langle a(x, u, \nabla u), \nabla u\rangle - \left\langle a(x, v, \nabla v), \nabla\left(\frac{u^p}{v^{p-1}}\right)\right\rangle &= \langle a(x, u, \nabla u), \nabla u\rangle \\ - p\left\langle a\left(x, v, \frac{u}{v}\nabla v\right), \nabla u\right\rangle &+ (p-1)\left\langle a\left(x, v, \frac{u}{v}\nabla v\right), \frac{u}{v}\nabla v\right\rangle.\end{aligned} \tag{1.4}$$

The proof of (1.4) is on the similar lines as the proof of (1.2). The aim of this paper is to establish several applications of (1.4) and for this purpose, let us consider the model problem

$$\begin{cases} -div(a(x, u, \nabla u)) = \lambda f(x, u, \nabla u) + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$