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Mechanics of affine bodies. Towards affine dynamical symmetry

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A B S T R A C T

In this paper we discuss certain dynamical models of affine bodies, including problems of partial separability and integrability. There are some reasons to expect that the suggested models are dynamically viable and that on the fundamental level of physical phenomena the "large" affine symmetry of dynamical laws is more justified and desirable than the restricted invariance under isometries.

1. Introduction

Below we present ideas and models which are rather new and up to our knowledge quite rarely touched in the literature (the exceptions are the Marsden's and Hughes' theorem in their book [14] and Mariano's theorem in [12]). In most papers we know (many of them quoted in the references [2,4,6,7,15,18,16,17,21,26–28]) it is only kinematics that is based on affine geometry, but on the dynamical level affine symmetry is broken and restricted to the Euclidean group of motions or some of its subgroups. Unlike this we discuss models the dynamics of which is affinely-invariant. For example we show that instead of using some explicit potential energy expression one can encode the dynamics of elastic vibrations in an appropriate form of affinely-invariant kinetic energy. The resulting geodetic models in a sense resemble the procedure of Maupertuis principle, where the orbits of motion are geodetics of an appropriate metric tensor built as some conformal modification of the "true" geometric fundamental tensor [1]. There is also some similarity with the concept of effective mass known from solid state physics [9]. In a sense, our affinely-invariant

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geodetic models may be interpreted as a discretization of the Arnold description of ideal fluids in terms of geodetic Hamiltonian systems on the group of volume-preserving diffeomorphisms [1]. This is a very drastic discretization, reducing the continuum cardinality of degrees of freedom to the finite one, namely $n(n+1)$, where n is denoting the dimension of the physical space ($n = 3$ in realistic models).

We mention about certain procedures based on modified finite elements methods [29] following the ideas of M. Rubin and his co-workers [19,20]. In its most traditional version the idea of finite elements was to cover the bulk of deformable body with a mesh of tetrahedrons or parallelepipeds (or sometimes other simple figures) of sufficiently small size; similarly, the surface was replaced by a mesh of triangles or parallelograms (or other simple two-dimensional figures). Those small parts were assumed to be (in a good approximation) homogeneously deformable. Basing on this assumption one replaced a system of partial differential equations by a discrete system of difference equations. And this approximation enabled one to perform directly the computer-aided numerical calculations. The time variable is then also discretized and its continuity is replaced by some mesh. But there is also another procedure, when instead of the spatio-temporal finite elements one uses also spatial ones. This hybrid method consists in an approximate representation of continuum as a system of mutually interacting affinely rigid (homogeneously deformable) bodies. But the time variable is continuous and for the mentioned system the usual methods of analytical mechanics and qualitative theory of dynamical systems are used. In various problems this hybrid discretized-analytical approach seems to be more appropriate. In any case the powerful methods of dynamical systems theory may be used.

At the end of the Introduction section let us shortly summarize the results that will be presented in this article. Later on we will develop some models which are ruled by affine group (either in the physical space or in the material) not only on the kinematical but also on the dynamical level. We may expect some physical applications of those models, e.g., in the collective nuclear dynamics, for description of defects in solids, in certain special problems in mechanics of deformable objects, and so on. The presented concept of affine bodies is obtained in the framework of elaboration of collective degrees of freedom and moments methods in many-body problems, mechanics of continuum media, and field theory, which is connected with classical discretization procedures (e.g., Rietz, Galerkin, etc.) and finite-elements methods. The possibility to encode elastic interactions without the usage of the potential term in the very kinetic energy (metric tensor on the configuration space), i.e., like in the Maupertuis variational principle, is illustrated through the two-polar decomposition of internal configurations and splitting the motion into the isochoric (incompressible) and dilatational parts. The formal analogy to the lattice-like structures after the partial diagonalization of the internal part of the affinely-invariant kinetic energy has been also discussed.

2. Some basic concepts

Although we admit discrete or even finite systems of material points, it is more convenient to use the standard terms of continua. Then configurations of continuous media are described by (even weak) diffeomorphisms Φ of the material space N ("Lagrange variables") onto the physical space M ("Euler variables") which are assumed to be affine spaces of the same dimension n (in applications $n = 3$, but sometimes also 2 or 1).

Obviously, motion is described by the time dependence of Φ. Usually (not always) M and N are endowed with flat metric tensors g, η in M, N; $g \in V^* \otimes V^*$, $\eta \in U^* \otimes U^*$ where V, U are respectively linear spaces of translations (free vectors) in M, N, and V^*, U^* are their duals (spaces of linear functions respectively on V, U). It is convenient to use rectangular coordinates a^K, y^i respectively in N and M. They are what is often referred to as Lagrange and Euler coordinates. The components of g, η are denoted by g_{ij}, η_{AB}; obviously in rectilinear rectangular coordinates we have $g_{ij} = \delta_{ij}$, $\eta_{AB} = \delta_{AB}$. The contravariant invariant metrics $g^{-1} \in V \otimes V$, $\eta^{-1} \in U \otimes U$ have components traditionally denoted by g^{ij}, η^{AB} (the upper case indices), where $g^{ik}g_{kj} = \delta^i{}_j$, $\eta^{AC}\eta_{CB} = \delta^A{}_B$.