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Identification of unstable fixed points for randomly perturbed dynamical systems with multistability

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ABSTRACT

Multistability, especially bistability, is one of the most important nonlinear phenomena in deterministic and stochastic dynamics. The identification of unstable fixed points for randomly perturbed dynamical systems with multistability has drawn increasing attention in recent years. In this paper, we provide a rigorous mathematical theory of the previously proposed data-driven method to identify the unstable fixed points of multistable systems. Specifically, we define a family of statistics which can be estimated by practical time-series data and prove that the local maxima of this family of statistics will converge to the unstable fixed points asymptotically. During the proof of the above result, we obtain two mathematical by-products which are interesting in their own right. We prove that the downhill timescale for randomly perturbed dynamical systems is $\log(1/\epsilon)$, different from the uphill timescale of $e^{V/\epsilon}$ for some $V > 0$ predicted by the Freidlin–Wentzell theory. Moreover, we also obtain an L^p maximum inequality for randomly perturbed dynamical systems and a class of diffusion processes.

1. Introduction

A number of deterministic and stochastic dynamical systems possess multiple stable or metastable equilibrium states. This phenomenon is widely referred to as multistability, which is one of the most important nonlinear phenomena in deterministic and stochastic dynamics [16]. Systems and devices with multistability, especially bistability, have been found or used in a wide range of scientific fields, including but not limited to mechanics, electronics, optics, thermodynamics, chemistry, biology, ecology, and meteorology. In the recent two decades, multistability has been extensively studied in biology. It has become increasingly clear that multistability is the key to understanding various basic cellular functions and the onset of complex diseases [20].

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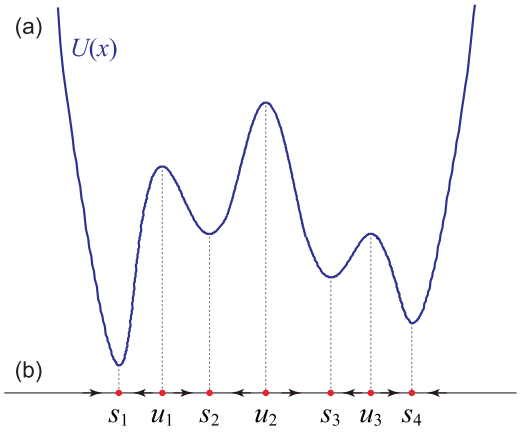

Fig. 1. (a) The potential of a multiscale system. (b) The phase portrait of the deterministic counterpart of the multiscale system.

Due to the stochastic effects, a multistable system in natural sciences is usually modeled by the following randomly perturbed dynamical system:

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \sqrt{\epsilon}\sigma(X_t^\epsilon)dB_t, \tag{1.1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. For simplicity, we only consider the one-dimensional case in this paper. A multistable system can be clearly described in terms of its potential, which is defined as

$$U(x) = -\int_0^x \frac{2b(y)}{\sigma^2(y)}dy. \tag{1.2}$$

The potential of a multistable system has multiple local minima and any two adjacent local minima are separated by a local maximum (see Fig. 1(a)). In the language of dynamical systems, the local minima and local maxima of the potential are the stable and unstable fixed points of the deterministic counterpart $\dot{x} = b(x)$ of the randomly perturbed dynamical system (1.1), respectively. Let s_i be all the stable fixed points and let u_i be all the unstable fixed points of the dynamical system $\dot{x} = b(x)$. Then s_i and u_i can be generally arranged as (see Fig. 1(b)):

$$-\infty < s_1 < u_1 < s_2 < \cdots < s_{k-1} < u_{k-1} < s_k < \infty.$$

In recent years, the identification of unstable fixed points for multistable systems has attracted increasing attention. Recent studies on complex diseases have shown that any disease progression can be divided into a normal state, a pre-disease state, and a disease state [3]. The normal and disease states correspond to the stable fixed points of a multistable system and the pre-disease state corresponds to the unstable fixed point between them. Once the expression level of the disease-related gene in a person is close to the unstable fixed point, we have good reasons to believe that this person is in a pre-disease state and is at high risk of disease progression. This suggests that the identification of unstable fixed points for multistable systems is closely related to the early diagnosis of complex diseases.

Now that the unstable fixed points of multistable systems are of great importance, it is natural to ask whether we can detect them in an effective way by using the experimental data. Recently, several research groups have proposed different methods to solve this problem [3,4,10,12]. In biological experiments, it often occurs that a large number of multistable systems with the same distribution can be observed or measured at several discrete times $t_1, t_2, \cdots, t_n$ with time interval $h(\epsilon) = t_{m+1} - t_m$. For example, the expression