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Weighted Bergman projections on the Hartogs triangle

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ABSTRACT

We prove the L^p regularity of the weighted Bergman projections on the Hartogs triangle, where the weights are powers of the distance to the singularity at the boundary. The restricted range of p is proved to be sharp. By using a two-weight inequality on the upper half plane with Muckenhoupt weights, we can consider a slightly wider class of weights.

1. Introduction

1.1. Setup

Let Ω be a domain in $\mathbb{C}^n$.

Definition 1.1. A measurable function μ is a *weight* on Ω, if $\mu > 0$ almost everywhere and is locally integrable on Ω.

For $p \geq 1$, we consider the weighted L^p space

$$L^p(\Omega, \mu) = \{f \text{ measurable on } \Omega : \|f\|_{L^p(\Omega,\mu)} < \infty\},$$

where $\|\cdot\|_{L^p(\Omega,\mu)}$ is the weighted L^p norm defined by

$$\|f\|_{L^p(\Omega,\mu)} = \left(\int_\Omega |f(z)|^p \, \mu(z) \, dV(z)\right)^{\frac{1}{p}}.$$

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Let $\mathcal{O}(\Omega)$ be the set of holomorphic functions on Ω. For $p = 2$, it is easy to see that, if μ is continuous and non-vanishing on Ω, then the analytic subspace $A^2(\Omega, \mu) = L^2(\Omega, \mu) \cap \mathcal{O}(\Omega)$ is closed in $L^2(\Omega, \mu)$.

Definition 1.2. For a continuous and non-vanishing weight μ on Ω, we define the *weighted Bergman projection* $\mathcal{B}_{\Omega,\mu}$ on Ω with the weight μ to be the orthogonal projection from $L^2(\Omega, \mu)$ to $A^2(\Omega, \mu)$. The weighted Bergman projection is an integral operator

$$\mathcal{B}_{\Omega,\mu}(f)(z) = \int_{\Omega} B_{\Omega,\mu}(z, \zeta) f(\zeta) \mu(\zeta)\, dV(\zeta),$$

where $B_{\Omega,\mu}(z, \zeta)$ is the *weighted Bergman kernel* with $(z, \zeta) \in \Omega \times \Omega$.

1.2. Results

In this paper, we study the L^p regularity of the weighted Bergman projection on the Hartogs triangle

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| < |z_2| < 1\}$$

with the weight

$$\mu(z) = |z_2|^{s'} |g(z_2)|^2, \tag{1.1}$$

where $z \in \mathbb{H}$, $s' \in \mathbb{R}$ and g is a non-vanishing holomorphic function on the unit disk $\mathbb{D}$. Note that on $\mathbb{H}$, $|z_2|$ is comparable to $|z|$.

We first consider the weight μ with $g \equiv 1$ in (1.1).

Theorem 1. *For $s' \in \mathbb{R}$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $\mathcal{B}_{\mathbb{H},s'}$ be the weighted Bergman projection on $\mathbb{H}$ with the weight $\mu(z) = |z_2|^{s'}$, where $z \in \mathbb{H}$.*

(1) *For $s' \in (-2, \infty)$, $\mathcal{B}_{\mathbb{H},s'}$ is L^p bounded if and only if $p \in \left(\frac{s+2k+4}{s+k+2}, \frac{s+2k+4}{k+2}\right)$.*
(2) *For $s' \in [-5, -2]$, $\mathcal{B}_{\mathbb{H},s'}$ is L^p bounded for $p \in (1, \infty)$.*
(3) *For $s' \in (-6, -5)$, then $k = -3$ and $s \in (0, 1)$, $\mathcal{B}_{\mathbb{H},s'}$ is L^p bounded if and only if $p \in \left(2 - s, \frac{2-s}{1-s}\right)$.*
(4) *When $s' = -6$, $\mathcal{B}_{\mathbb{H},s'}$ is L^p bounded for $p \in (1, \infty)$.*
(5) *For $s' \in (-\infty, -6)$, $\mathcal{B}_{\mathbb{H},s'}$ is L^p bounded if and only if $p \in \left(\frac{s+2k+4}{k+2}, \frac{s+2k+4}{s+k+2}\right)$.*

Remark 1.3. A similar result holds for the n-dimensional generalization of the Hartogs triangle. See section 3 for details.

Remark 1.4. We point out that in Theorem 1 the range of p does not change continuously as s' varies. It is rather surprising that when s' is slightly larger than -2, the range of p is just a small neighborhood of 2. Whereas at $s' = -2$, we have the full range $p \in (1, \infty)$. In fact, similar jumps happen at the right hand sides of all the even integers, except at $s' = -4$. The reason is that the analytic subspace $A^2\big(\mathbb{H}, |z_2|^{s'}\big)$ remains fixed as long as s' does not go past the even integers.

To consider a wider class of weights of the form in (1.1), inspired by the ideas in [11,18], we use a different method and prove the following result.