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Asymptotic behavior of solutions to a class of non-autonomous delay differential equations

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we study the asymptotic behavior of solutions of a class of non-autonomous delay differential equations. These equations have important practical applications and generalize those on which Bernfeld and Haddock conjectured that each solution of the equations tends to a constant. It is shown that every solution of the equations is bounded and tends to a constant as $t \to +\infty$, which improves and extends some existing ones.

1. Introduction

Recently, the scalar delay differential equation,

$$x'(t) = -F(x(t)) + F(x(t-r)), \tag{1.1}$$

has been extensively studied (see, for example, [1,5–8,10–12,16–19] and the references cited therein) because of their applications in modeling population growth, the spread of epidemics, the dynamics of capital stocks and so on. When $F(x) = x^{\frac{1}{3}}$, Bernfeld and Haddock [4] conjectured that each solution of (1.1) tends to a constant. Furthermore, it was shown in the above mentioned references that each solution of equation (1.1) tends to a constant as $t \to +\infty$ under the assumption that F is either strictly increasing or locally Lipschitz and nondecreasing. Though the uniqueness of the left-hand solution of the following differential equation,

$$\begin{cases} x'(t) = -F(x(t)) + F(c), \\ x(t_0) = x_0 \quad \text{for all } t_0, x_0 \in \mathbf{R}, \end{cases} \tag{1.2}$$

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played a crucial role in the discussion of [5,6,10–12], this is not true as demonstrated by a counterexample in [20]. Consequently, in order to correct the proofs in [5,6,10–12], Ding adopted the following additional assumption.

(H) If $c \neq 0$, then the left-hand solution of the equation (1.2) is unique.

See the Appendix of [20].

It should be mentioned that it is difficult to provide some sufficient conditions guaranteeing the uniqueness of the left-hand solution of the initial value problem (1.2). So far it has not been proved whether $F(x) = x^{\frac{1}{3}}$ satisfies (H) or not. Therefore, the proof in the appendix of [20] needs further improvement. This is also true for the paper [21] as it extended the results in [5,6,10–12]. On the other hand, delays in population and ecology models are usually time-varying and hence these models are generalized to be described by non-autonomous functional differential equations. As a result, we can generalize the equations in the Bernfeld–Haddock conjecture to the following non-autonomous delay differential equations,

$$x'(t) = \gamma(t)[-x^{\frac{1}{n}}(t) + x^{\frac{1}{n}}(t - \tau(t))], \tag{1.3}$$

where $\tau(t)$ and $\gamma(t)$ are all continuous functions and are bounded above and below by positive constants, and n is a positive odd number. Obviously, the non-autonomous pantograph equation in [2,3,9,13,14] is only a special case of (1.3) with $n = 1$.

Motivated by the above discussion, we aim to employ a novel argument to provide some sufficient conditions which guarantee the uniqueness of the left-hand solution of the initial value problem (1.2), which can be used to show that every solution of (1.3) tends to a constant as $t \to +\infty$. Throughout this paper, we let $r = \sup_{t\in\mathbf{R}} \tau(t) \geq \inf_{t\in\mathbf{R}} \tau(t) > 0$ and $C = C([-r, 0], \mathbf{R})$. If $\sigma \geq 0$, $t_0 \in \mathbf{R}$, and $x \in C([t_0 - r, t_0 + \sigma], \mathbf{R})$, then, for any $t \in [t_0, t_0 + \sigma]$, $x_t \in C$ is defined by $x_t(t_0, \theta) = x(t_0, t + \theta)$, $-r \leq \theta \leq 0$. Moreover, for $\varphi \in C$, we use $x_t(t_0, \varphi)$ $(x(t; t_0, \varphi))$ to denote the solution of (1.3) with the initial data $x_{t_0}(t_0, \varphi) = \varphi$. For $V(t) \in C([a, \infty),\ \mathbf{R})$, let

$$D^+V(t) = \limsup_{h\to 0^+} \frac{V(t+h) - V(t)}{h} \quad \text{and} \quad D^-V(t) = \liminf_{h\to 0^+} \frac{V(t+h) - V(t)}{h}.$$

The remaining part of this paper is organized as follows. In Section 2, we recall some relevant results, and give a detailed proof on the uniqueness of the left-hand solution of the initial value problem (1.2) with $F(x) = qx^{\frac{1}{n}}$. Meanwhile, we show the boundedness and global existence of every solution for (1.3) with the initial data $x_{t_0} = \varphi \in C$. Based on the preparation in Section 2, we state and prove our main result in Section 3. In Section 4, we give some examples to illustrate the effectiveness of the obtained results by numerical simulation. We also formulate some relevant open problems.

2. Preliminary results

Assume that $F : \mathbf{R} \to \mathbf{R}$ is continuous and strictly increasing. Then, from Proposition 4* and Proposition 5* in [20], we have

Proposition 2.1. *Let* (H) *hold. Consider the differential equation,*

$$u' = -F(u) + F(c + \varepsilon), \tag{2.1}$$

where $c \neq 0$ is a given constant, ε is a parameter satisfying $0 \leq \varepsilon \leq \frac{|c|}{2}$. Moreover, assume the initial condition