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The Floquet–Bloch transform and scattering from locally perturbed periodic surfaces

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A R T I C L E I N F O

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A B S T R A C T

We use the Floquet–Bloch transform to reduce variational formulations of surface scattering problems for the Helmholtz equation from periodic and locally perturbed periodic surfaces to equivalent variational problems formulated on bounded domains. To this end, we establish various mapping properties of that transform between suitable weighted Sobolev spaces on periodic strip-like domains and coupled families of quasiperiodic Sobolev spaces. Our analysis shows in particular that the decay of solutions to surface scattering problems from locally perturbed periodic surfaces is precisely characterized by the smoothness of its Bloch transform in the quasiperiodicity.

1. Introduction

We analyze time-harmonic surface scattering modeled by the scalar Helmholtz equation from a periodic or a locally perturbed periodic surface Γ. A fundamental motivation to study surface scattering problems involving periodicity is the growing industrial importance of micro or nano-structured surfaces in optics, requiring ever more accurate models and simulations. Neglecting periodicity, such scattering problems can be considered as rather particular rough surface scattering problems and tackled by variational or integral equation formulations [5,4,6,3]. Unfortunately, these formulations are posed on unbounded domains such that, e.g., convergence analysis for any numerical discretization automatically is non-standard.

Motivated by the recent paper [7] which treats the Helmholtz equation in a periodic medium with a line defect by the (Floquet–)Bloch transform, we show in this paper that an analogous partial transform can be used to transform scattering problems for unbounded periodic surfaces with local perturbations to equivalent problems on bounded domains. Any variational formulation of such a transformed problems hence possesses the advantage of straightforward discretization by standard techniques. Note that [12] analyzes the discretization of such a problem in two dimensions in a setting that is somewhat easier due to absorption, and that [11] uses the Bloch transform to study scattering in an infinite wave guide.

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The Floquet–Bloch transform can be interpreted as, roughly speaking, a sort of periodic Fourier transform, see [17,13]. Despite mapping properties of that transform are in principle known, in particular its isometry property on L^2-spaces, we prove that a certain partial Bloch transform is an isomorphism between weighted Sobolev spaces on periodic surfaces or periodic strips and coupled families of Sobolev spaces of quasiperiodic functions. (Suitable references seem to be lacking.) Even if these mapping properties are independent of dimension, we restrict ourselves to surfaces and domains of dimension two and three, respectively, see Remark 1. (See further [10, Annexe B] for corresponding results in the one-dimensional case.)

The partial Bloch transform allows to equivalently reformulate scattering problems from periodic or locally perturbed periodic surfaces as a (coupled) family of quasiperiodic scattering problems on, roughly speaking, a unit cell of the periodic domain. Apart from the Bloch transform, our main technique for treating perturbed periodic surfaces is a suitable diffeomorphism between the perturbed periodic domain and the unperturbed one. We exploit this equivalence to analyze solutions to scattering problems for particular incident Herglotz wave functions that may serve, e.g., as models for incident Gaussian beams. For simplicity, we restrict ourselves to Robin or impedance boundary conditions involving a periodic coefficient, and merely comment on other boundary conditions.

Our results in particular show that the decay of the solution to a scattering problem from a (perturbed) periodic surface is characterized by smoothness of the solution to the transformed problem in the quasiperiodicity. Decay results for solutions to rough surface scattering problems with Dirichlet boundary condition have been established previously in [3]; some of these results are, roughly speaking, validated by our findings for different boundary conditions (see Remark 14). We further indicate that for a particular class of incident Herglotz wave functions, the wave field scattered from a (locally perturbed) periodic surface decays more rapidly than the bounds from [3] would suggest.

The remainder of this paper is structured as follows: After presenting the Bloch transform in $\mathbb{R}^2$ in Section 2, we introduce function spaces to analyze this transform in Section 3. Next, Sections 4, 5, and 6 show properties of the Bloch transform on $\mathbb{R}^2$, on (bi-)periodic surfaces, and on periodic domains, respectively. Section 7 introduces surface scattering problems of incident acoustic waves from periodic surfaces and tackles those by the Bloch transform. Section 8 extends these results by regularity estimates for the solution in the quasiperiodicity parameter. Section 9 finally tackles similar problems for perturbed periodic surface scattering via suitable diffeomorphisms.

Notation: We write $\widetilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \boldsymbol{x}_2)^\top$ for points $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)^\top$ in $\mathbb{R}^3$ and set, in analogy, $\widetilde{\nabla} f = (\partial_{\boldsymbol{x}_1} f, \partial_{\boldsymbol{x}_2} f)^\top$, whereas $\nabla f = (\partial_{\boldsymbol{x}_1} f, \partial_{\boldsymbol{x}_2} f, \partial_{\boldsymbol{x}_3} f)^\top$ is the gradient of f. By $\mathbf{e}_{(1,2,3)}$ we denote the standard basis vectors of $\mathbb{R}^3$. All function spaces we consider generically contain complex-valued functions. The space of smooth functions in a domain U with smooth extension to the boundary up to arbitrarily high order is $C^\infty(\overline{U})$. Constants C and c are generic and might change from line to line. We further use the symbol $\simeq$ to indicate equivalence of two expressions up to positive constants.

2. The Bloch transform in $\mathbb{R}^2$

In the entire paper we fix an invertible matrix $\Lambda \in \mathbb{R}^{2\times 2}$ and call a function or a vector field $\varphi : \mathbb{R}^2 \to \mathbb{C}^d$ Λ-periodic if $\varphi(\boldsymbol{x} + \Lambda j) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^2$ and all $j \in \mathbb{Z}^2$. It satisfies to require the latter condition in the fundamental domain of periodicity for the lattice $\{\Lambda \boldsymbol{j} : \boldsymbol{j} \in \mathbb{Z}^2\} \subset \mathbb{R}^2$, the so-called Wigner–Seitz-cell,

$$W_\Lambda := \left\{\Lambda \widetilde{z} : \widetilde{\boldsymbol{z}} \in \mathbb{R}^2,\ -1/2 < \widetilde{\boldsymbol{z}}_{1,2} \leq 1/2\right\} \subset \mathbb{R}^2.$$

The dual periodicity matrix then equals $\Lambda^* := 2\pi \Lambda^{-\top} = 2\pi(\Lambda^\top)^{-1} \in \mathbb{R}^{2\times 2}$ and defines the dual (or reciprocal) lattice $\{\Lambda^* \boldsymbol{\ell} : \boldsymbol{\ell} \in \mathbb{Z}^2\}$, as well as the dual fundamental domain of periodicity, the so-called Brillouin zone

$$W_{\Lambda^*} := \left\{\Lambda^* \widetilde{\boldsymbol{\mu}} : \widetilde{\boldsymbol{\mu}} \in \mathbb{R}^2,\ -1/2 < \widetilde{\boldsymbol{\mu}}_{1,2} \leq 1/2\right\} \subset \mathbb{R}^2.$$