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Operational solution for some types of second order differential equations and for relevant physical problems

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ABSTRACT

We present an operational method to obtain solutions for differential equations, describing a broad range of physical problems, including ordinary non-integer order and high order partial differential equations. Inverse differential operators are proposed to solve a variety of differential equations. Integral transforms and the operational exponent are used to obtain the solutions. Generalized families of orthogonal polynomials and special functions are also employed with recourse to their operational definitions. Examples of solutions of physical problems, related to propagation of the heat and other quantities are demonstrated by the developed operational technique. In particular, the evolution type problems, the generalizations of the Black–Scholes, of the heat conduction, of the Fokker–Planck equations are considered as well as equations, involving the Laguerre derivative operator.

1. Introduction

Differential equations play very important role in mathematics and their role in the description of physical processes cannot be overestimated. Thus, obtaining the solutions for differential equations is of paramount importance. Few types of differential equations (DE) allow explicit and straightforward analytical solutions. Lately, development of computer methods facilitated equations solving. However, understanding of the obtained solutions and their application to description of physical processes in some cases is still challenging. Despite the revolutionary breakthrough in computer methods in the 21st century, analytical studies remain requested because, in general, they allow transparent meaning of the solutions obtained. Indeed, expansion in series of orthogonal polynomials [1] is useful for solving many physical problems (see, for example, [15] and [16]). Hermite, Laguerre and others polynomial families are usually defined by series or sums; for the purpose of differential equations solution and in the context of the operational approach they are best defined by operational relations. These polynomials possess generalized forms with many variables and

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indices [4,10], which arise naturally in studies of physical problems, related to propagation and radiation of accelerated charges, heat and mass transfer and other phenomena (see, for example, [8,12,25,30]). The analytical description of the influence of additional non-periodic magnetic components in undulator radiation (UR) [6,24,26,27] involves generalized Airy and Bessel type functions, which are also related to generalized forms of orthogonal polynomials. In what follows, we develop an analytical method to obtain the solutions for various types of differential equations on the base of the operational identities, employing special functions, expansions in the series of Hermite, Laguerre polynomials and their modified forms [1,13]. Some examples of the relevant physical problems will be considered in what follows. The mathematical instruments mentioned above are very useful for solution of broad range of physical problems. Recent developments in technique and technology in synchrotron radiation (SR) and undulator radiation (UR) induced particular interest to the analysis of the radiation from charged particle beams and their propagation in accelerators and in insertion devices. Free electron lasers (FEL) open new frontiers for researcher, but they also require high quality radiation sources and high precision undulators. This calls for modern mathematical instruments, suitable for the analysis of the undergoing physical processes and for the adequate description of the performance of the devices. Analytical solutions have this value since they usually possess more transparent physical meaning than numerical solutions and allow deep understanding of the processes. When it comes to a numerical analysis, there are also practical and theoretical reasons for examining the process of inverting differential operators. Indeed, the inverse or integral form of a differential equation displays explicitly the input–output relationship of the system. Moreover, integral operators are computationally and theoretically less troublesome than differential operators; for example, differentiation emphasizes data errors, whereas integration averages them. Thus, it may be advantageous to apply computational procedures to differential systems, based on the inverse or integral description of the system.

The concept of an inverse function is just a function that undoes another function. That is if an input x into the function f yields an output y, then putting y into the inverse function g produces the output x, and vice versa. i.e., $f(x) = y$ and $g(y) = x$ or $g(f(x)) = x$. If a function f has an inverse f^{-1}, it is invertible and the inverse function is then uniquely determined by f. We can develop similar approach with regard to differential operators. In what follows we shall develop the operational approach and explore its relation with extended forms of orthogonal polynomials to obtain analytical solutions for a broad class of differential equations, including evolution type equations, generalized forms of heat, mass transfer and Black–Scholes type equations, involving also the Laguerre derivative operator.

For a general form of differential equation

$$\psi(D)F(x) = f(x), \tag{1}$$

where $\psi(D)$ is a differential operator, the inverse differential operator $1/\psi(D)$ or $(\psi(D))^{-1}$ is such that

$$\psi(D)(\psi(D))^{-1}f(x) = f(x). \tag{2}$$

From (1) we obtain the particular integral

$$F(x) = (\psi(D))^{-1}f(x). \tag{3}$$

It is easy to prove the following identity:

$$(\psi(D))^{-1}e^{\alpha x}f(x) = e^{\alpha x}(\psi(D+\alpha))^{-1}f(x), \tag{4}$$

and the action of the inverse operator $(\psi(D+\alpha))^{-1}$ on a function f, which can be expressed via the inverse differential operator $(\psi(D))^{-1}$, reads as follows:

$$F(x) = (\psi(D+\alpha))^{-1}f(x) = e^{-\alpha x}(\psi(D))^{-1}e^{\alpha x}f(x). \tag{5}$$