J. Math. Anal. Appl. 446 (2017) 663–680

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multivariate delta Gončarov and Abel polynomials

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ARTICLE INFO

Article history:
Received 2 June 2016
Available online 7 September 2016
Submitted by B.C. Berndt

Keywords:
Abel and Gončarov polynomials
Appell relations
Delta operators
Interpolation
Umbral calculus

ABSTRACT

Classical Gončarov polynomials are polynomials which interpolate derivatives. Delta Gončarov polynomials are polynomials which interpolate delta operators, e.g., forward and backward difference operators. We extend fundamental aspects of the theory of classical bivariate Gončarov polynomials and univariate delta Gončarov polynomials to the multivariate setting using umbral calculus. After introducing systems of delta operators, we define multivariate delta Gončarov polynomials, show that the associated interpolation problem is always solvable, and derive a generating function (an Appell relation) for them. We show that systems of delta Gončarov polynomials on an interpolation grid $Z \subseteq \mathbb{R}^d$ are of binomial type if and only if $Z = A\mathbb{N}^d$ for some $d \times d$ matrix A. This motivates our definition of delta Abel polynomials to be exactly those delta Gončarov polynomials which are based on such a grid. Finally, compact formulas for delta Abel polynomials in all dimensions are given for separable systems of delta operators. This recovers a former result for classical bivariate Abel polynomials and extends previous partial results for classical trivariate Abel polynomials to all dimensions.

1. Introduction

The main purpose of this paper is to extend some fundamental aspects of the theory of Gončarov and Abel polynomials to higher dimensions and to replace partial derivatives by systems of delta operators.

Such topics have an extensive history. In 1881, Abel [1] introduced a sequence $g_0, g_1, g_2, \ldots$ of polynomials, now carrying his name, to represent analytic functions. These polynomials are determined by the condition that they interpolate the derivatives of *any* given analytic function $f : \mathbb{R} \to \mathbb{R}$ at the nodes of an arithmetic progression through the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{g_n(x)}{n!} f^{(n)}(nb), \tag{1}$$

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http://dx.doi.org/10.1016/j.jmaa.2016.09.002
0022-247X/

where $b \in \mathbb{R}$ is a fixed parameter. In particular, it can be shown that $g_n(x) = x(x - nb)^{n-1}$ for every n, and g_n satisfies the orthogonality condition

$$g_n^{(k)}(kb) = n!\delta_{k,n} \qquad \text{for all } k.$$

Abel polynomials count some basic combinatorial objects, for example, labeled trees and (generalized) parking functions [24]. The sequence $\{g_n(x)\}_{n\geq 0}$ is of binomial type, which leads to a connection with umbral calculus (or finite operator calculus), a branch of mathematics that studies analytic and algebraic-combinatorial properties of polynomials by a systematic use of operator methods.

The (classical) umbral calculus of polynomials in one variable was put onto a firm theoretical basis by Rota et al. in a series of papers [16,21,22]. It was extended to multivariate polynomials and applied to combinatorial problems in [2,17–19]. In [3,7,23], higher dimensional umbral calculus is used to derive various versions of the Lagrange inversion formula.

The operators considered in umbral calculus are delta operators, a family of linear operators, acting on the algebra of univariate polynomials with coefficients in a field. Delta operators share many properties in common with derivatives. Each delta operator $\mathfrak{d}$ is uniquely associated with a sequence $\{p_n\}$ of polynomials of binomial type, which interpolates the iterates of $\mathfrak{d}$, evaluated at 0, as

$$f(x) = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} [\mathfrak{d}^n f(x)]_{x=0}.$$

In [4,5], Gončarov allowed the interpolation grid to be arbitrary, obtaining that

$$f(x) = \sum_{n=0}^{\infty} g_n(x; a_0, a_1, \ldots, a_n) f^{(n)}(a_n), \tag{2}$$

where $a_n \in \mathbb{R}$ and $g_n(x; a_0, a_1, \ldots, a_n)$ are the Gončarov polynomials. Such polynomials have an extensive history in numerical analysis, even for interpolation of derivatives [14]. Gončarov polynomials in two or more variables have the unusual property that the interpolation problem is solvable for any choice of the nodes of interpolation. The uniqueness was shown in [12] for the bivariate case and in [6] for the multivariate case.

It is well known that certain values of (univariate) Gončarov polynomials are connected with order statistics (e.g., see [9]). This connection has been further developed [11] into a complete correspondence between Gončarov polynomials and parking functions, a discrete structure lying at the heart of combinatorics. In [10], difference Gončarov polynomials were studied. Since difference operators are delta operators, this was another connection with umbral calculus.

Our basic goal here is to extend the theory of classical multivariate Gončarov polynomials by using umbral calculus and replacing partial derivatives with *delta operators*. This is a further development of our previous work [8,15] on the analytic and combinatorial properties of bivariate Gončarov polynomials and [13] on the interpolation with general univariate delta operators. In addition, the theory of generating functions, polynomial recursion and approximation theory (the latter being Abel's original motivation for studying his Abel polynomials), just to mention a few, all play a role here.

The rest of the paper is organized as follows. Section 2 contains the definition and basic properties of a system of delta operators in d variables. In Section 3 we define the multivariate Gončarov polynomials associated with a system of delta operators and an interpolation grid Z, derive a generating function (Appell relation), and characterize the set of delta Abel polynomials, which are multivariate Gončarov polynomials of binomial type associated with delta operators. In the last two sections, we present closed formulas for multivariate delta Abel polynomials in the special case when these are associated with a *separable* system of delta operators. In particular, Section 4 deals with the bivariate case, and Section 5 contains the general formulas in an arbitrary dimension.