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On critical systems involving fractional Laplacian ☆

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ABSTRACT

Consider the following non-local critical system

$$\begin{cases}(-\Delta)^s u-\lambda_1 u=\mu_1|u|^{2_*-2}u+\frac{\alpha\gamma}{2_*}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega,\\ (-\Delta)^s v-\lambda_2 v=\mu_2|v|^{2_*-2}v+\frac{\beta\gamma}{2_*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega,\\ u=0,\ v=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases} \tag{0.1}$$

where $(-\Delta)^s$ is fractional Laplacian, $0<s<1$ and all $\lambda_1,\lambda_2,\mu_1,\mu_2,\gamma>0$, $2_*:=\frac{2N}{N-2s}$ is a fractional Sobolev critical exponent, $N>2s$, $\alpha,\beta>1$, $\alpha+\beta=2_*$, and Ω is an open bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Under proper conditions, we establish the existence result of the ground state solution to system (0.1).

1. Introduction

Recently, fractional Sobolev spaces and the corresponding nonlocal equations are applied to many subjects, such as, among others, anomalous diffusion, elliptic problems with measure data, gradient potential theory, minimal surfaces, non-uniformly elliptic problems, optimization, phase transitions, quasigeostrophic flows, singular set of minima of variational functionals, and water waves (see [4] and the references therein). For fractional Laplacian, we refer to [1,7–12]. Single equation involving fractional Laplacian had been investigated by some researchers, such as

$$\begin{cases}(-\Delta)^s u-\lambda u=|u|^{2_*-2}u & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

which was studied in [1,11].

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In present paper, we consider the following critical system with fractional Laplacian:

$$\begin{cases}(-\Delta)^s u-\lambda_1 u=\mu_1|u|^{2_*-2}u+\frac{\alpha\gamma}{2_*}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega,\\ (-\Delta)^s v-\lambda_2 v=\mu_2|v|^{2_*-2}v+\frac{\beta\gamma}{2_*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega,\\ u=0,\ v=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases} \tag{1.1}$$

where $0<s<1$ and all $\lambda_1,\lambda_2,\mu_1,\mu_2,\gamma$ are positive, $2_*:=\frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $N>2s$, $\alpha,\beta>1$, $\alpha+\beta=2_*$; Ω is an open bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and $-(-\Delta)^s$ is fractional Laplacian defined by

$$-(-\Delta)^s u(x)=\frac{C(N,s)}{2}\int\limits_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y,\quad x\in\mathbb{R}^N,$$

where

$$C(N,s)=\left(\int\limits_{\mathbb{R}^N}\frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\mathrm{d}\zeta\right)^{-1}=2^{2s}\pi^{-\frac{N}{2}}\frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(2-s)}s(1-s). \tag{1.2}$$

Define the Hilbert space $D_s(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{D_s}$ induced by the scalar product $\langle\cdot,\cdot\rangle_{D_s}$ given by

$$\langle u,v\rangle_{D_s}:=\frac{C(N,s)}{2}\int\limits_{\mathbb{R}^{2N}}\frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

If Ω is an open bounded Lipschitz domain, then $D_s(\Omega)$ coincides with the Sobolev space

$$X_0:=\{f\in X: f=0 \text{ a.e. in } \Omega^c\},$$

where X is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function f in X belongs to $L^2(\Omega)$ and the map $(x,y)\mapsto\big(f(x)-f(y)\big)|x-y|^{-\frac{N}{2}+s}$ is in $L^2\left(\mathbb{R}^{2N}\setminus(\Omega^c\times\Omega^c),\ \mathrm{d}x\mathrm{d}y\right)$, and Ω^c is the complement of Ω in $\mathbb{R}^N$. Usually, there are two ways to define fractional Sobolev space. One is via Gagliardo seminorm

$$H^s(\mathbb{R}^N):=\left\{u\in L^2(\mathbb{R}^N):\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}}\in L^2(\mathbb{R}^{2N})\right\},$$

the other is via Fourier transformation

$$\hat{H}^s(\mathbb{R}^N):=\left\{u\in L^2(\mathbb{R}^N):\int\limits_{\mathbb{R}^N}\left(1+|\xi|^{2s}\right)\left|\mathcal{F}u(\xi)\right|^2\mathrm{d}\xi<+\infty\right\},$$

and $H^s(\mathbb{R}^N)=\hat{H}^s(\mathbb{R}^N)$. In present paper, we choose the one via Gagliardo seminorm

$$\begin{aligned}[u]^2_{H^s(\mathbb{R}^N)}:=&\frac{C(N,s)}{2}\int\limits_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ =&\int\limits_{\mathbb{R}^N}|\xi|^{2s}\left|\mathcal{F}u(\xi)\right|^2\mathrm{d}\xi.\end{aligned}$$