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Global strong solutions for 3D viscous incompressible heat conducting magnetohydrodynamic flows with non-negative density

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ABSTRACT

We study an initial boundary value problem for the nonhomogeneous heat conducting magnetohydrodynamic fluids with non-negative density. Firstly, it is shown that for the initial density allowing vacuum, the strong solution to the problem exists globally if the gradients of velocity and magnetic field satisfy $\|\nabla \mathbf{u}\|_{L^4(0,T;L^2)} + \|\nabla \mathbf{b}\|_{L^4(0,T;L^2)} < \infty$. Then, under some smallness condition, we prove that there is a unique global strong solution to the 3D viscous incompressible heat conducting magnetohydrodynamic flows. Our method relies upon the delicate energy estimates and regularity properties of Stokes system and elliptic equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth simply connected domain, the motion of a viscous, incompressible, and heat conducting magnetohydrodynamic fluid in Ω can be described by the following MHD system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(2\mu \mathfrak{D}(\mathbf{u})) + \nabla P = \mathbf{b} \cdot \nabla \mathbf{b}, \\ c_v[\partial_t(\rho\theta) + \operatorname{div}(\rho \mathbf{u}\theta)] - \kappa \Delta \theta = 2\mu |\mathfrak{D}(\mathbf{u})|^2 + \nu |\operatorname{curl} \mathbf{b}|^2, \\ \partial_t \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{b} = \nu \Delta \mathbf{b}, \\ \operatorname{div} \mathbf{u} = 0, \ \operatorname{div} \mathbf{b} = 0 \end{cases} \tag{1.1}$$

with the initial condition

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$$(\rho, \mathbf{u}, \theta, \mathbf{b})(0, x) = (\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary condition

$$\mathbf{u} = \mathbf{0}, \ \frac{\partial \theta}{\partial \mathbf{n}} = 0, \ \mathbf{b} \cdot \mathbf{n} = 0, \ \operatorname{curl} \mathbf{b} \times \mathbf{n} = \mathbf{0}, \ \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Here, $t \geq 0$ is time, $x \in \Omega$ is the spatial coordinate, and ρ, $\mathbf{u}$, θ, P, $\mathbf{b}$ are the fluid density, velocity, absolute temperature, pressure, and the magnetic field, respectively; $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{tr}).$$

The constant $\mu > 0$ is the viscosity coefficient. Positive constants c_v and κ are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field.

Magnetohydrodynamics studies the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. The issues of well-posedness and dynamical behaviors of MHD system are rather complicated to investigate because of the strong coupling and interplay interaction between the fluid motion and the magnetic field. In the absence of magnetic field, that is, $\mathbf{b} = \mathbf{0}$, the MHD system reduces to the Navier–Stokes equations. In general, due to the similarity of the second equation and the third equation in (1.1), the study for MHD system has been along with that for Navier–Stokes one.

When $\mathbf{b} = \mathbf{0}$ and $\theta = 0$, the system (1.1) reduces to the well-known nonhomogeneous incompressible Navier–Stokes equations and there are a lot of results on the existence in the literature. In the case that the viscosity μ is a constant, Kazhikov [16] (see also [2]) proved that when ρ_0 is bounded away from zero, the nonhomogeneous Navier–Stokes equations have at least one global weak solution in the energy space. In addition, he also proved the global existence of strong solutions to this system for small data in three space dimensions and all data in two dimensions. When the initial data may contain vacuum states, Simon [25] obtained the global existence of weak solutions, and Choe–Kim [6] proposed a compatibility condition and investigated the local existence of strong solutions, which was later improved by Craig–Huang–Wang [7] for global strong small solutions. On the other hand, in the case that μ depends on the density ρ, Lions [20, Chapter 2] established the global existence of weak solutions to nonhomogeneous Navier–Stokes equations in any space dimensions for the initial density allowing vacuum. Recently, Cho–Kim [4] proved the local existence of unique strong solutions for initial data satisfying a natural compatibility condition. Very recently, Huang–Wang [15], and independently by Zhang [29], showed the global existence of strong solutions on bounded domains under some smallness assumption. There are also very interesting investigations about the existence of strong solutions to the 2D nonhomogeneous Navier–Stokes equations, refer to [14,19,21,24].

For the study of nonhomogeneous incompressible MHD system (i.e., $\theta = 0$ in (1.1)), Gerbeau and Le Bris [11], Desjardins and Le Bris [8] studied the global existence of weak solutions with finite energy on 3D bounded domains and on the torus, respectively. In the presence of vacuum, under the following compatibility conditions,

$$\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0, \ -\mu\Delta\mathbf{u}_0 + \nabla P_0 - (\mathbf{b}_0 \cdot \nabla)\mathbf{b}_0 = \sqrt{\rho_0}\mathbf{g}, \quad \text{in } \Omega, \tag{1.4}$$

where $(P_0, \mathbf{g}) \in H^1 \times L^2$ and $\Omega = \mathbb{R}^3$, Chen–Tan–Wang [3] obtained the local existence of strong solutions to the 3D Cauchy problem of (1.1). Later, Gong–Li [12] showed that the local strong solution obtained in