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# Polynomial solutions of the Hermitian submonogenic system

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ABSTRACT

The Hermitian monogenic system is an overdetermined system of two Dirac type operators in several complex variables generalizing both the holomorphic system and the real Dirac system. Due to the fact that it is overdetermined, the Cauchy–Kowalevski extension problem only has a solution if the Cauchy data satisfy certain constraints. There is however a subsystem, called Hermitian submonogenic system, for which theses constraints are no longer necessary, while, if the constraints hold, the Cauchy–Kowalevski extension will still be Hermitian monogenic. In this paper we focus on Cauchy–Kowalevski extensions of general polynomials, in the case of the Hermitian submonogenic system, and we compute the corresponding dimensions and reproducing kernels.



## 1. Introduction

Hermitian Clifford analysis is a very rich and well-established field of research (see e.g. [1,2,5,14]). It deals with the so-called Hermitian monogenic functions, which are in the kernel of two mutually adjoint Dirac operators invariant under the action of the unitary group. It is interesting to note that by suitably restricting the values of the Hermitian monogenic functions one obtains holomorphic functions in several complex variables. Hermitian Clifford analysis is also a refinement of the more classical Clifford analysis dealing with functions in the kernel of the Dirac operator, which is invariant under the action of the special orthogonal group. For a panorama on the research in this field we refer the reader to [15] and to the references therein.

The Cauchy–Kowalevski extension theorem for the Hermitian monogenic system has been considered in [4], but for a complete treatment of this problem and to solve it for any given analytic function, it is better to introduce a weaker system called the Hermitian submonogenic system (see [10,11]). The Hermitian submonogenic system shows a better behavior, compared with the Hermitian monogenic system, also from

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other points of view. For example, as we shall see in Section 2, its null solutions coincide with the kernel of just one operator $\mathbb{D}$ which factorizes a product of two Laplace operators. Moreover, the only monogenic functions which are Hermitian submonogenic are the Hermitian monogenic functions.

Motivated by these facts, we continue in this paper the study of the Hermitian submonogenic system which started in [10]. There the Cauchy–Kowalevski extension problem was studied and some special solutions were also determined. Furthermore, a Cauchy kernel and an integral representation formula have recently been obtained in [7]. The main purpose of this paper is to study the polynomial solutions of the system.

We start by recalling some basic notations and facts on Clifford algebras and on some classes of monogenic functions which will be useful in the sequel.

Let $\mathbb{R}_{0,m}$ denote the real Clifford algebra generated by the standard basis $\{e_1, \ldots, e_m\}$ of the Euclidean space $\mathbb{R}^m$ (see [6]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, \ldots, m$$

and a general element $a \in \mathbb{R}_{0,m}$ may be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

in terms of the basis elements $e_A = e_{j_1} \cdots e_{j_k}$, defined for every subset $A = \{j_1, \ldots, j_k\}$ of $\{1, \ldots, m\}$ with $j_1 < \cdots < j_k$.

For the empty set one puts $e_\emptyset = 1$, the latter being the identity element. Conjugation in $\mathbb{R}_{0,m}$ is given by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{j_k} \cdots \overline{e}_{j_1}$ with $\overline{e}_j = -e_j$, $j = 1, \ldots, m$.

The fundamental first order differential operator in $\mathbb{R}^m$ given by $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ is called the Dirac operator, and its null solutions are called monogenic functions (see e.g. [3,8,9,12,13]).

**Definition 1.** A function $F : \Omega \to \mathbb{R}_{0,m}$ defined and continuously differentiable in an open set $\Omega \subset \mathbb{R}^m$, is said to be (left) monogenic if

$$\partial_{\underline{x}} F = 0 \text{ in } \Omega.$$

The complex Clifford algebra $\mathbb{C}_m$ is defined as $\mathbb{C}_m = \mathbb{R}_{0,m} \oplus i\,\mathbb{R}_{0,m}$. Therefore, any complex Clifford number $c \in \mathbb{C}_m$ has the form $c = a + ib$, where $a, b \in \mathbb{R}_{0,m}$. The Hermitian conjugate of $c$ is defined to be $c^\dagger = \overline{a} - i\overline{b}$.

The consideration of complexified Clifford algebras over even dimensional spaces leads to the construction of the so-called Witt basis of $\mathbb{C}_m$. Let us assume that $m = 2n$ and define the so-called Witt basis

$$\mathfrak{f}_j = \frac{1}{2}(e_j - ie_{n+j}), \quad \mathfrak{f}_j^\dagger = -\frac{1}{2}(e_j + ie_{n+j}), \quad j = 1, \ldots, n.$$

They satisfy the Grassmann identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0$$

as well as the duality identities

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk}, \quad j, k = 1, \ldots, n.$$

Using the Witt basis we can rewrite the Clifford vector $\underline{x} = \sum_{j=1}^n (x_j e_j + y_j e_{n+j})$ as

$$\underline{x} = \sum_{j=1}^n \left(z_j \mathfrak{f}_j - \overline{z}_j \mathfrak{f}_j^\dagger\right),$$