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Regularization and derivatives of multipole potentials

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ABSTRACT

The only harmonic homogeneous functions defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ are the harmonic polynomials and the so-called multipole potentials, namely functions of the type $P(\mathbf{x}) = p(\mathbf{x}) / |\mathbf{x}|^{2k+n-2}$ for some harmonic polynomial p of degree k. The first aim of this article is to study the distributional regularization of multipole potentials. We show that even though the Hadamard regularization $\mathcal{P}f\left(p(\mathbf{x}) / |\mathbf{x}|^{2k+n-2}\right)$ exists for any homogeneous polynomial of degree k, the principal value p.v. $\left(p(\mathbf{x}) / |\mathbf{x}|^{2k+n-2}\right)$ exists if and only if p is harmonic; this means that if p is harmonic then for any test function ϕ the *divergent* integral $\int_{\mathbb{R}^n} p(\mathbf{x}) \phi(\mathbf{x}) / |\mathbf{x}|^{2k+n-2} \, \mathrm{d}\mathbf{x}$ can be computed by employing polar coordinates and performing the angular integral first. We also find the first and second order distributional derivatives of these regularizations and, more generally, of the regularizations of functions of the form $P_l(\mathbf{x}) = p(\mathbf{x}) / |\mathbf{x}|^{k+l}$. We find many interesting formulas that hold precisely when p is a harmonic polynomial of degree k. In particular, we prove that

$$\overline{\Delta}\text{p.v.}\left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right) = \frac{(-1)^{k+1}\pi^{n/2}}{2^{k-2}\Gamma\left(\frac{n}{2}+k-1\right)} p(\boldsymbol{\nabla})\,\delta(\mathbf{x}) \ ,$$

generalizing the well known relation $\overline{\Delta}\left(r^{2-n}\right) = (2-n)\, C\delta(\mathbf{x})$, where C is the area of a sphere of radius 1. Actually formulas like this one hold for a homogeneous polynomial p of degree k if and only if p is harmonic.

1. Introduction

There are some functions that are both harmonic and homogeneous, $u(\lambda\mathbf{x}) = \lambda^{\alpha} u(\mathbf{x})$, $\lambda > 0$. In the whole space $\mathbb{R}^n$ the only possibility is $\alpha = k \in \mathbb{N}$, and in that case u must be a polynomial function, $u \in \mathcal{H}_k$, where we denote by $\mathcal{H}_k$ the set of harmonic homogeneous functions of degree k. Actually one may consider $\mathcal{H}_k$ under three different lights, namely, as a set of polynomials in n variables of degree k, or as a set of polynomial *functions*, perhaps better denoted as $\mathcal{H}_k(\mathbb{R}^n)$, or even as the set of restrictions to the unit

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sphere, $\mathcal{H}_k(\mathbb{S})$. The elements of $\mathcal{H}_k(\mathbb{S})$ are usually called spherical harmonics, while those of $\mathcal{H}_k(\mathbb{R}^n)$ are referred to as solid harmonics. Notice that the restriction map $\mathcal{H}_k(\mathbb{R}^n) \longrightarrow \mathcal{H}_k(\mathbb{S})$ is a bijection because of the maximum principle for harmonic functions. See [1] for the many properties of harmonic polynomials and of harmonic functions in general.

On the other hand, we may consider harmonic homogeneous functions defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. One way to obtain such functions is to apply the Kelvin transform [1, Chp. 4], $u \mapsto K[u]$ to the elements $u \in \mathcal{H}_k$. In general, if u is a function defined in a region $\Omega \subset \mathbb{R}^n$, then $v = K[u]$ is a function defined in the conjugated set $\Omega^* = \{\mathbf{x}^* : \mathbf{x} \in \Omega\}$, $\mathbf{x}^* = \mathbf{x}/|\mathbf{x}|^2$, by $v(\mathbf{x}) = |\mathbf{x}|^{2-n} u(\mathbf{x}^*)$; the Kelvin transform sends harmonic functions to harmonic functions and it also sends homogeneous functions to homogeneous functions, so that if $p \in \mathcal{H}_k$ then the function

$$P(\mathbf{x}) = \frac{p(\mathbf{x})}{r^{2k+n-2}}, \tag{1.1}$$

where, as customary, $r = |\mathbf{x}|$, is a harmonic function, homogeneous of degree $-k-n+2$, defined[1] in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Functions of the form (1.1) are sometimes called *multipole potentials* [18]. It is not hard to see that all harmonic homogeneous functions defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ are either harmonic polynomials or multipole potentials of the form (1.1).

The aim of this article is to study several properties of multipole potentials and, more generally, of functions of the type $P_l(\mathbf{x}) = r^{-k-l}p(\mathbf{x})$ where $p \in \mathcal{H}_k$. Since $P = P_{k+n-2}$ has a non-integrable singularity at the origin, unless $k = 0$ or $k = 1$, we need to study the *regularization* of P as a distribution of the space $\mathcal{D}'(\mathbb{R}^n)$; we find that the principal value distribution $\text{p.v.}(P(\mathbf{x})) \in \mathcal{D}'(\mathbb{R}^n)$ given as

$$\langle \text{p.v.}(P(\mathbf{x})), \phi(\mathbf{x})\rangle = \lim_{\varepsilon \to 0} \int\limits_{|\mathbf{x}| \geq \varepsilon} P(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \phi \in \mathcal{D}(\mathbb{R}^n), \tag{1.2}$$

always exists if $p \in \mathcal{H}_k$. This means that if $p \in \mathcal{H}_k$ one may regularize the *divergent* integral $\int_{\mathbb{R}^n} P(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}$ by following the simple rule of "*using polar coordinates and performing the angular integrals first.*" In fact, we show that if p is a general homogeneous polynomial of degree k which is not harmonic, then the principal value does not exist, so that the simple rule of regularization *does not work* and one needs to employ the Hadamard regularization $\mathcal{P}f(P(\mathbf{x}))$.

Next we obtain formulas for the first and second order derivatives of the distributions $\mathcal{P}f(P_l(\mathbf{x}))$, in particular for the distributions $\text{p.v.}(P_{k+n-2}(\mathbf{x}))$. Such derivatives are very important in Mathematical Physics [3,4,15,18] and several special cases have been computed by several authors [13,17,21,22]. Naturally it is rather simple to obtain the ordinary derivatives of $P_l(\mathbf{x})$, that is, the derivatives away from the origin,[2] therefore we pay special attention to the *delta part* of these derivatives. Our computations show that many times the derivatives of fields that do not have a delta part may have a high order delta part, that is, derivatives of the delta function can appear in the derivatives of fields that have no delta function at the origin, as (1.3) already shows; this "apparent paradox" was pointed out by Parker [22], who warns of the mistakes that it can produce.

The distributional derivatives of any order of power potentials were given in [8,9], and are available in several textbooks [10,20], and in principle could be employed to compute the distributional derivatives of $\mathcal{P}f(P_l(\mathbf{x}))$, even if p is not harmonic, but such direct computations become rather complicated and no simple formulas are obtained. Nevertheless, we show that when $p \in \mathcal{H}_k$ the expressions for the derivatives can be simplified in a surprising way, leading to rather nice formulas. In particular we show that[3]

1 In fact P is defined in $\widetilde{\mathbb{R}^n} \setminus \{\mathbf{0}\}$, the one point compactification of $\mathbb{R}^n$, $\widetilde{\mathbb{R}^n}$, with the origin removed.
2 In other words, this is the far field behavior.
3 An overbar denotes a distributional derivative, a notation first introduced by the late Professor Farassat [11].